

$\mathcal{N}(p, q, s)$ -TYPE SPACES IN THE UNIT BALL OF \mathbb{C}_n BINGYANG HU AND SONGXIAO LI[†]

ABSTRACT. In this paper, we introduce a new class of space, called $\mathcal{N}(p, q, s)$ -type spaces, in the unit ball \mathbb{B} . We study some basic properties, Hadamard gaps, Hadamard products, atomic decomposition of $\mathcal{N}(p, q, s)$ -type spaces. Moreover, we also establish several equivalent characterizations, including Carleson measure characterization and various derivative characterizations. Finally, we also characterize the distance between Bergman-type spaces and $\mathcal{N}(p, q, s)$ -type spaces.

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1. INTRODUCTION

Let \mathbb{B} be the open unit ball in \mathbb{C}^n with \mathbb{S} as its boundary and $H(\mathbb{B})$ the collection of all holomorphic functions in \mathbb{B} . H^∞ denotes the Banach space consisting of all bounded holomorphic functions in \mathbb{B} with

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the norm $\|f\|_\infty = \sup_{z \in \mathbb{B}} |f(z)|$. For $l > 0$, the Bergman-type space $A^{-l}(\mathbb{B})$ is the space of all $f \in H(\mathbb{B})$ such that

$$|f|_l = \sup_{z \in \mathbb{B}} |f(z)|(1 - |z|^2)^l < \infty.$$

Let $A_0^{-l}(\mathbb{B})$ denote the closed subspace of $A^{-l}(\mathbb{B})$ such that $\lim_{|z| \rightarrow 1} |f(z)|(1 - |z|^2)^l = 0$.

Denote dV the normalized volume measure over \mathbb{B} and for $\alpha > -1$, the weighted Lebesgue measure dV_α is defined by

$$dV_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dV(z)$$

where

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)}$$

is a normalizing constant so that dV_α is a probability measure on \mathbb{B} .

For $\alpha > -1$ and $p > 0$ the weighted Bergman space A_α^p consists of holomorphic functions f in \mathbb{B} satisfying

$$\|f\|_{p,\alpha} = \left[\int_{\mathbb{B}} |f(z)|^p dV_\alpha(z) \right]^{1/p} < \infty.$$

It is well-known that when $1 \leq p < \infty$, A_α^p is a Banach space and when $0 < p < 1$, A_α^p becomes a complete metric space. We refer the reader to the excellent book [25] for more information.

Let $\Phi_a(z)$ be the automorphism of \mathbb{B} for $a \in \mathbb{B}$, i.e.,

$$\Phi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a \rangle},$$

where $s_a = \sqrt{1 - |a|^2}$, P_a is the orthogonal projection into the space spanned by a and $Q_a = I - P_a$ (see, e.g., [25]). The \mathcal{N}_p -space on \mathbb{B} was introduced in [4], i.e., for $p > 0$,

$$\begin{aligned} \mathcal{N}_p &= \mathcal{N}_p(\mathbb{B}) \\ &= \left\{ f \in H(\mathbb{B}) : \|f\|_p = \sup_{a \in \mathbb{B}} \left(\int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \right)^{1/2} < \infty \right\}. \end{aligned}$$

The little space of \mathcal{N}_p -space, denoted by \mathcal{N}_p^0 , which consists of all $f \in \mathcal{N}_p$ such that

$$\lim_{|a| \rightarrow 1^-} \int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) = 0.$$

Let $d\lambda(z) = \frac{dV(z)}{(1-|z|^2)^{n+1}}$, then $d\lambda$ is Möbius invariant (see, e.g., [18]), which means,

$$\int_{\mathbb{B}} f(z) d\lambda(z) = \int_{\mathbb{B}} f \circ \phi(z) d\lambda(z)$$

for each $f \in L^1(\lambda)$ and ϕ an automorphism of \mathbb{B} .

A Banach space of complex functions on a set X is called a *functional Banach space* on X if the vector operations are the pointwise operations, $f(x) = g(x)$ for each x in X implies $f = g$, $f(x) = f(y)$ for each function f in the space implies $x = y$, and for each x in X , the point evaluation $K_x : f \mapsto f(x)$ is a continuous linear functional.

For $p \geq 1$ and $q, s > 0$, we introduce the $\mathcal{N}(p, q, s)$ -type space as follows:

$$\begin{aligned} & \mathcal{N}(p, q, s) \\ & := \left\{ f \in H(\mathbb{B}) : \|f\|^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty \right\}, \end{aligned}$$

with the corresponding little space

$$\begin{aligned} & \mathcal{N}^0(p, q, s) \\ & := \left\{ f \in \mathcal{N}(p, q, s) : \lim_{|a| \rightarrow 1^-} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) = 0 \right\}. \end{aligned}$$

It should be noted that when $p = 2, q = n + 1, s > 0$, $\mathcal{N}(2, n + 1, s)$ coincides the \mathcal{N}_{ns} -space, as well as the little space, in particular, by [4, Theorem 2.1], when $p = 2, q = n + 1, s > 1$, $\mathcal{N}(2, n + 1, s) = A^{-\frac{n+1}{2}}(\mathbb{B})$. Moreover, we note that $\mathcal{N}(p, q, s)$ -type spaces are Möbius invariant while the \mathcal{N}_p -spaces are not.

The aim of the present paper is to study various properties of $\mathcal{N}(p, q, s)$ -type spaces. We will characterize some basic properties, Hadamard gaps, Hadamard products, atomic decomposition of $\mathcal{N}(p, q, s)$ -type spaces. We also establish several equivalent characterizations, including Carleson measure characterization and various derivative characterizations. Finally, we will investigate the distance between Bergman-type spaces and $\mathcal{N}(p, q, s)$ -type spaces.

Throughout this paper, for $a, b \in \mathbb{R}$, $a \lesssim b$ ($a \gtrsim b$, respectively) means there exists a positive number C , which is independent of a and b , such that $a \leq Cb$ ($a \geq Cb$, respectively). Moreover, if both $a \lesssim b$ and $a \gtrsim b$ hold, then we say $a \simeq b$.

2. BASIC PROPERTIES OF $\mathcal{N}(p, q, s)$ -TYPE SPACES

2.1. Basic structure. We first show that $\mathcal{N}(p, q, s)$ is a functional Banach space. For $a \in \mathbb{B}$ and $0 < R < 1$, define

$$D(a, R) = \Phi_a(\{z \in \mathbb{B} : |z| < R\}) = \{z \in \mathbb{B} : |\Phi_a(z)| < R\}.$$

The following result plays an important role in the sequel.

Proposition 2.1. *Let $p \geq 1$ and $q, s > 0$. The point evaluation $K_z : f \mapsto f(z)$ is a continuous linear functional on $\mathcal{N}(p, q, s)$. Moreover, $\mathcal{N}(p, q, s) \subseteq A^{-\frac{q}{p}}(\mathbb{B})$.*

Proof. Denote $\mathbb{B}_{1/2} := \{z : |z| < \frac{1}{2}\}$. For each $f \in \mathcal{N}(p, q, s)$ and $a_0 \in \mathbb{B}$, we have

$$\begin{aligned} \|f\|^p &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\ &\geq \int_{D(a_0, 1/2)} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_{a_0}(z)|^2)^{ns} d\lambda(z) \\ &\geq C_{n,s} \int_{D(a_0, 1/2)} |f(z)|^p (1 - |z|^2)^q d\lambda(z) \\ &= C_{n,s} \int_{\mathbb{B}_{1/2}} |f(\Phi_{a_0}(w))|^p (1 - |\Phi_{a_0}(w)|^2)^q d\lambda(w) \\ &\quad (\text{change variable } z = \Phi_{a_0}(w)) \\ &= C_{n,s} \int_{\mathbb{B}_{1/2}} |f(\Phi_{a_0}(w))|^p \frac{(1 - |a_0|^2)^q (1 - |w|^2)^{q-n-1}}{|1 - \langle a_0, w \rangle|^{2q}} dV(w) \\ &\geq C_{q,s,n} (1 - |a_0|^2)^q \int_{\mathbb{B}_{1/2}} |f(\Phi_{a_0}(w))|^p dV(w) \\ &\geq C'_{q,s,n} (1 - |a_0|^2)^q |f(a_0)|^p \quad (|f \circ \Phi_{a_0}(\cdot)|^p \text{ is subharmonic.}), \end{aligned}$$

where $C_{n,s}$ is a constant depending on n and s and $C_{q,s,n}$ and $C'_{q,s,n}$ are some constants depending on q, s and n . Hence, for any $z \in \mathbb{B}$, we have

$$|f(z)| \lesssim \frac{\|f\|}{(1 - |z|^2)^{\frac{q}{p}}},$$

which implies the point evaluation is a continuous linear functional, as well as, $\mathcal{N}(p, q, s) \subseteq A^{-\frac{q}{p}}(\mathbb{B})$. \square

Letting $q = n + 1$ and $p = 2$ in Proposition 2.1, we get [4, Theorem 2.1, (a)] as a particular case.

Corollary 2.2. *For $p > 0$, $\mathcal{N}_p(\mathbb{B}) \subseteq A^{-\frac{n+1}{2}}(\mathbb{B})$.*

Theorem 2.3. *Let $p \geq 1$ and $q, s > 0$. $\mathcal{N}(p, q, s)$ is a functional Banach space.*

Proof. It is clear that $\mathcal{N}(p, q, s)$ is a normed vector space with respect to the norm $\|\cdot\|$. It suffices to show the completeness of $\mathcal{N}(p, q, s)$. Let $\{f_m\}$ be a Cauchy sequence in $\mathcal{N}(p, q, s)$. From this, by Proposition 2.1, it follows that $\{f_m\}$ is a Cauchy sequence in the space $H(\mathbb{B})$, and hence it converges to some $f \in H(\mathbb{B})$. It remains to show that $f \in \mathcal{N}(p, q, s)$. Indeed, there exists a $\ell_0 \in \mathbb{N}$ such that for all $m, \ell \geq \ell_0$, it holds $\|f_m - f_\ell\| \leq 1$. Take and fix an arbitrary $a \in \mathbb{B}$, by Fatou's lemma, we have

$$\begin{aligned} & \int_{\mathbb{B}} |f(z) - f_{\ell_0}(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\ & \leq \lim_{\ell \rightarrow \infty} \int_{\mathbb{B}} |f_\ell(z) - f_{\ell_0}(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\ & \leq \lim_{\ell \rightarrow \infty} \|f_\ell - f_{\ell_0}\|^p \leq 1, \end{aligned}$$

which implies that

$$\|f - f_{\ell_0}\| = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z) - f_{\ell_0}(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \leq 1,$$

and hence $\|f\| \leq 1 + \|f_{\ell_0}\| < \infty$. Thus, combining this with the fact that the point evaluation on $\mathcal{N}(p, q, s)$ is a continuous linear functional, we conclude that $\mathcal{N}(p, q, s)$ is a functional Banach space. \square

Next we show that when $p \geq 1$, $q > 0$ and $s > 1$, $\mathcal{N}(p, q, s) = A^{-\frac{q}{p}}(\mathbb{B})$. More precisely, we have the following result.

Proposition 2.4. *Let $p \geq 1$, $q, s > 0$. If $s > 1 - \frac{q-kp}{n}$, $k \in \left(0, \frac{q}{p}\right]$, then $A^{-k}(\mathbb{B}) \subseteq \mathcal{N}(p, q, s)$. In particular, when $s > 1$, $\mathcal{N}(p, q, s) = A^{-\frac{q}{p}}(\mathbb{B})$.*

Proof. Suppose $p \geq 1$, $q > 0$ and $s > \frac{q-kp}{n}$ for some $k \in \left(0, \frac{q}{p}\right]$. Since $s > \frac{q-kp}{n}$, we have $q + ns - n - 1 - kp > -1$, then by [25, Theorem 1.12], for each $a \in \mathbb{B}$, we have

$$\int_{\mathbb{B}} \frac{(1 - |z|^2)^{q+ns-n-1-kp}}{|1 - \langle a, z \rangle|^{2ns}} dV(z) \simeq \begin{cases} \text{bounded in } \mathbb{B}, & \text{if } ns + pk < q; \\ \log \frac{1}{1-|a|^2}, & \text{if } ns + pk = q; \\ (1 - |z|^2)^{q+ns-pk}, & \text{if } ns + pk > q, \end{cases}$$

which implies, there exists a positive constant C such that

$$(2.1) \quad \sup_{a \in \mathbb{B}} (1 - |a|^2)^{ns} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{q+ns-n-1-kp}}{|1 - \langle a, z \rangle|^{2ns}} dV(z) \leq C.$$

Let $f \in A^{-k}(\mathbb{B})$, by (2.1), we have

$$\begin{aligned}
\|f\|^p &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\
&= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^{pk} (1 - |z|^2)^{q-pk} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\
&\leq |f|_k^p \sup_{a \in \mathbb{B}} (1 - |a|^2)^{ns} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{q+ns-n-1-pk}}{|1 - \langle a, z \rangle|^{2ns}} dV(z) \\
&\leq C |f|_k^p,
\end{aligned}$$

which implies $A^{-k}(\mathbb{B}) \subseteq \mathcal{N}(p, q, s)$.

Now if $s > 1$, then in particular, we can take $k = \frac{q}{p}$ and hence by the above argument, we have $A^{-\frac{q}{p}}(\mathbb{B}) \subseteq \mathcal{N}(p, q, s)$. Combing this fact with Proposition 2.1, we get the desired result. \square

Letting $q = n + 1$ and $p = 2$ in Proposition 2.4, we get [4, Theorem 2.1, (b)] as a particular case.

2.2. The closure of all polynomials in $\mathcal{N}(p, q, s)$ -type spaces. We have the following description of $\mathcal{N}^0(p, q, s)$.

Proposition 2.5. *Let $p \geq 1$ and $q, s > 0$. Then $\mathcal{N}^0(p, q, s)$ is a closed subspace of $\mathcal{N}(p, q, s)$ and hence $\mathcal{N}^0(p, q, s)$ is a Banach space.*

Proof. First we note that it is trivial to see that $\mathcal{N}^0(p, q, s)$ is a subspace of $\mathcal{N}(p, q, s)$ and hence it suffice to show that $\mathcal{N}^0(p, q, s)$ is complete. Suppose $\{f_n\}$ is a Cauchy sequence in $\mathcal{N}^0(p, q, s)$, by Theorem 2.3, there is a limit $f \in \mathcal{N}(p, q, s)$ of $\{f_n\}$. For any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$, such that when $n > N$, $\|f - f_n\| < (\frac{\varepsilon}{2^p})^{1/p}$. Take $n_0 > N$, since $f_{n_0} \in \mathcal{N}^0(p, q, s)$, there exists a $\delta \in (0, 1)$ such that when $\delta < |a| < 1$,

$$\sup_{\delta < |a| < 1} \int_{\mathbb{B}} |f_{n_0}(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \frac{\varepsilon}{2^p}.$$

Hence, we have

$$\begin{aligned}
&\sup_{\delta < |a| < 1} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\
&\leq 2^{p-1} \|f - f_{n_0}\|^p \\
&\quad + 2^{p-1} \sup_{\delta < |a| < 1} \int_{\mathbb{B}} |f_{n_0}(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \varepsilon.
\end{aligned}$$

which implies that $f \in \mathcal{N}^0(p, q, s)$. \square

Theorem 2.6. *Suppose $f \in \mathcal{N}(p, q, s)$ with $ns + q > n$. Then $f \in \mathcal{N}^0(p, q, s)$ if and only if $\|f_r - f\| \rightarrow 0$, $r \rightarrow 1^-$, where $f_r(z) = f(rz)$ for all $z \in \mathbb{B}$.*

Proof. • **Necessity.** Suppose $f \in \mathcal{N}^0(p, q, s)$. This implies that for any $\varepsilon > 0$, there exists a $\delta > 0$, such that with $\delta < |a| < 1$, we have

$$(2.2) \quad \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \frac{\varepsilon}{3 \cdot 2^{2n+p-1}}.$$

Furthermore, by Schwarz-Pick Lemma (see, e.g., [18, Theorem 8.1.4]), we have

$$(2.3) \quad |\Phi_{ra}(rz)| \leq |\Phi_a(z)|, \text{ for all } r \in (0, 1) \text{ and } a, z \in \mathbb{B}.$$

Now take and fix $\delta_0 \in (\delta, 1)$. Consider r satisfying $\max\left\{\frac{1}{2}, \frac{\delta}{\delta_0}\right\} < r < 1$. In this case, for all $a \in \mathbb{B}$ with $|a| \in (\delta_0, 1)$, by (2.2) and (2.3), we have

$$\begin{aligned} & \int_{\mathbb{B}} |f(rz)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\ & \leq \int_{\mathbb{B}} |f(rz)|^p (1 - |rz|^2)^q (1 - |\Phi_{ra}(rz)|^2)^{ns} d\lambda(z) \\ & = \left(\frac{1}{r}\right)^{2n} \int_{\mathbb{B}} |f(w)|^p (1 - |w|^2)^q (1 - |\Phi_{ra}(w)|^2)^{ns} d\lambda(w) \\ & \leq 4^n \int_{\mathbb{B}} |f(w)|^p (1 - |w|^2)^q (1 - |\Phi_{ra}(w)|^2)^{ns} d\lambda(w) < \frac{\varepsilon}{3 \cdot 2^{p-1}}. \end{aligned}$$

On the other hand, since $ns + q > n$, we have $ns + q - n - 1 > -1$. By [25, Propostion 2.6], f_r converges to f as $r \rightarrow 1^-$, in the norm topology of the Bergman space $A_{ns+q-n-1}^p(\mathbb{B})$. This implies that there exists a $r_1 \in (0, 1)$, such that for $r_1 < r < 1$, we have

$$\int_{\mathbb{B}} |f(rz) - f(z)|^p (1 - |z|^2)^{ns+q-n-q} dV(z) < \frac{(1 - \delta_0)^{2ns} \cdot \varepsilon}{3}.$$

Hence, for $|a| \leq \delta_0$ and $r_1 < r < 1$, we have

$$\begin{aligned} & \sup_{|a| \leq \delta_0} \int_{\mathbb{B}} |f(rz) - f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\ & = \sup_{|a| \leq \delta_0} \left\{ (1 - |a|^2)^{ns} \int_{\mathbb{B}} |f(rz) - f(z)|^p \frac{(1 - |z|^2)^{ns+q-n-1}}{|1 - \langle a, z \rangle|^{2ns}} dV(z) \right\} \\ & \leq \sup_{|a| \leq \delta_0} \int_{\mathbb{B}} |f(rz) - f(z)|^p \frac{(1 - |z|^2)^{ns+q-n-1}}{|1 - \langle a, z \rangle|^{2ns}} dV(z) \\ & \leq \frac{1}{(1 - \delta_0)^{2ns}} \int_{\mathbb{B}} |f(rz) - f(z)|^p (1 - |z|^2)^{ns+q-n-1} dV(z) < \frac{\varepsilon}{3}. \end{aligned}$$

Consequently, for all r with $\max\left\{\frac{1}{2}, \frac{\delta}{\delta_0}, r_1\right\} < r < 1$, we have

$$\begin{aligned}
\|f_r - f\|^p &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(rz) - f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\
&\leq \left(\sup_{|a| \leq \delta_0} + \sup_{\delta_0 < |a| < 1} \right) \int_{\mathbb{B}} |f(rz) - f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\
&\leq \frac{\varepsilon}{3} + 2^{p-1} \sup_{\delta_0 < |a| < 1} \int_{\mathbb{B}} (|f(rz)|^p + |f(z)|^p) (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\
&< \frac{\varepsilon}{3} + 2^{p-1} \left(\frac{\varepsilon}{3 \cdot 2^{p-1}} + \frac{\varepsilon}{3 \cdot 2^{2n+p-1}} \right) \\
&< \frac{\varepsilon}{3} + 2^{p-1} \left(\frac{\varepsilon}{3 \cdot 2^{p-1}} + \frac{\varepsilon}{3 \cdot 2^{p-1}} \right) = \varepsilon,
\end{aligned}$$

which shows that $\|f_r - f\| \rightarrow 0$ as $r \rightarrow 1^-$.

• **Sufficiency.** First we show that for each $r \in (0, 1)$ and each $f \in \mathcal{N}(p, q, s)$, then $f_r \in \mathcal{N}^0(p, q, s)$. By [25, Theorem 1.12], we have

$$\int_{\mathbb{B}} \frac{(1 - |z|^2)^{q+ns-n-1}}{|1 - \langle a, z \rangle|^{2ns}} dV(z) \simeq \begin{cases} \text{bounded in } \mathbb{B}, & \text{if } ns < q; \\ \log \frac{1}{1-|a|^2}, & \text{if } ns = q; \\ (1 - |a|^2)^{q-ns}, & \text{if } ns > q, \end{cases}$$

which implies

$$\sup_{a \in \mathbb{B}} (1 - |a|^2)^{ns} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{q+ns-n-1}}{|1 - \langle a, z \rangle|^{2ns}} dV(z) < \infty.$$

Hence, we have

$$\begin{aligned}
\|f_r\|^p &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f_r(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\
&\leq \left(\sup_{z \in \mathbb{B}} |f(rz)| \right)^p \cdot \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\
&= \left(\sup_{z \in \mathbb{B}} |f(rz)| \right)^p \cdot \sup_{a \in \mathbb{B}} (1 - |a|^2)^{ns} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{q+ns-n-1}}{|1 - \langle a, z \rangle|^{2ns}} dV(z),
\end{aligned}$$

which is finite. Thus, $f \in \mathcal{N}(p, q, s)$. Moreover, we also have

$$\begin{aligned}
&\int_{\mathbb{B}} |f_r(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\
&\leq M \int_{\mathbb{B}} (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z),
\end{aligned}$$

for some $M > 0$ (this is due to the boundedness of the function $f(z)$ on $\{|z| \leq r\}$).

Noting that $ns + q > n$ and by the estimation on the term

$\int_{\mathbb{B}} \frac{(1-|z|^2)^{q+ns-n-1}}{|1-\langle a, z \rangle|^{2ns}} dV(z)$ above, we have

$$\int_{\mathbb{B}} (1-|z|^2)^q (1-|\Phi_a(z)|^2)^{ns} d\lambda(z) \rightarrow 0, \text{ as } |a| \rightarrow 1^-,$$

which implies that $f_r \in \mathcal{N}^0(p, q, s)$ for all $r \in (0, 1)$.

Now suppose that $\|f_r - f\| \rightarrow 0$ as $r \rightarrow 1^-$. Then by the fact that $\mathcal{N}^0(p, q, s)$ is a closed subspace of $\mathcal{N}(p, q, s)$, it follows that $f \in \mathcal{N}^0(p, q, s)$. \square

As a corollary of Theorem 2.6, we obtain the following result.

Corollary 2.7. *The set of polynomials is dense in $\mathcal{N}^0(p, q, s)$ when $ns + q > n$.*

Proof. By Theorem 2.6, for any $f \in \mathcal{N}^0(p, q, s)$, we have

$$\lim_{r \rightarrow 1^-} \|f_r - f\| = 0.$$

Since each f_r can be uniformly approximated by polynomials, and moreover, by the proof of Theorem 2.6, the sup-norm norm in \mathbb{B} dominates the $\mathcal{N}(p, q, s)$ -norm, we conclude that every $f \in \mathcal{N}^0(p, q, s)$ can be approximated in the $\mathcal{N}(p, q, s)$ -norm by polynomials. \square

However, generally, it is not true that $\mathcal{N}(p, q, s)$ contains all the polynomials. Precisely, we have the following result.

Proposition 2.8. *Let $p \geq 1$ and $q, s > 0$. Then the set of polynomials are contained in $\mathcal{N}(p, q, s)$ if and only if $ns + q > n$.*

Proof. The sufficiency clearly follows from Theorem 2.6 and Corollary 2.7. Next we prove the necessity. Assume $ns + q \leq n$ and denote $\alpha = n - ns - q, \alpha \geq 0$. We claim that in this case, the constant function $F(z) = 1, z \in \mathbb{B}$ does not belong to $\mathcal{N}(p, q, s)$. Indeed,

$$\begin{aligned} \|F\|^p &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} (1-|z|^2)^q (1-|\Phi_a(z)|^2)^{ns} d\lambda(z) \\ &= \sup_{a \in \mathbb{B}} (1-|a|^2)^{ns} \int_{\mathbb{B}} \frac{(1-|z|^2)^{q+ns-n-1}}{|1-\langle a, z \rangle|^{2ns}} dV(z) \\ &\geq \int_{\mathbb{B}} \frac{1}{(1-|z|^2)^{\alpha+1}} dV(z) \quad (\text{Put } a = 0) \\ &\simeq \int_0^1 \frac{r^{2n-1}}{(1-r^2)^{1+\alpha}} dr \gtrsim \int_{1/2}^1 \frac{1}{(1-r)^{1+\alpha}} dr \\ &= \infty, \end{aligned}$$

which is a contradiction. Hence, we get the desired result. \square

2.3. Description given by Green's function. In [21] and [22], the invariant Green's function is defined as $G(z, a) = g(\Phi_a(z))$, where

$$g(z) = \frac{n+1}{2n} \int_{|z|}^1 (1-t^2)^{n-1} t^{-2n+1} dt.$$

The following property of g is important (see, e.g., [15]).

Proposition 2.9. *Let $n \geq 2$ be an integer. Then there are positive constants C_1 and C_2 such that for all $z \in \mathbb{B} \setminus \{0\}$,*

$$C_1(1 - |z|^2)^n |z|^{-2(n-1)} \leq g(z) \leq C_2(1 - |z|^2)^n |z|^{-2(n-1)}.$$

For $p \geq 1, q, s > 0$ and $n \geq 2$, we define the following $\mathcal{N}_*(p, q, s)$ -type space:

$$\begin{aligned} \mathcal{N}_*(p, q, s) \\ := \left\{ f \in H(\mathbb{B}) : \|f\|_*^p = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) < \infty \right\}, \end{aligned}$$

where the corresponding little space is defined as

$$\begin{aligned} \mathcal{N}_*^0(p, q, s) \\ := \left\{ f \in \mathcal{N}_*(p, q, s) : \lim_{|a| \rightarrow 1^-} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) = 0 \right\}. \end{aligned}$$

Noting that by Proposition 2.9, it is clear that $\|\cdot\| \lesssim \|\cdot\|_*$, that is, $\mathcal{N}_*(p, q, s) \subseteq \mathcal{N}(p, q, s)$. Combining this fact with the proof of Proposition 2.1, Theorem 2.3 and Proposition 2.5, we have the following result.

Theorem 2.10. *For $p \geq 1, q, s > 0$ and $n \geq 2$. $\mathcal{N}_*(p, q, s)$ is a functional Banach space and $\mathcal{N}_*^0(p, q, s)$ is a closed subspace of $\mathcal{N}_*(p, q, s)$. Moreover, $\mathcal{N}_*(p, q, s) \subseteq A^{-\frac{q}{p}}(\mathbb{B})$.*

However, generally, it is not true that $\mathcal{N}_*(p, q, s)$ contain all the polynomials, for example, by Proposition 2.8 and the fact that $\|\cdot\| \lesssim \|\cdot\|_*$, we know that when $ns + q \leq n$, $F \notin \mathcal{N}_*(p, q, s)$, where $F(z) = 1, z \in \mathbb{B}$. We are interested in the following question: when does $\mathcal{N}_*(p, q, s)$ contain the set of polynomials?

We have the following result.

Proposition 2.11. *Let $p \geq 1, q, s > 0$ and $n \geq 2$. Then the set of polynomials are contained in $\mathcal{N}_*(p, q, s)$ if and only if $ns + q > n$ and $s < \frac{n}{n-1}$.*

Proof. • **Necessity.** We prove it by contradiction. Consider the constant function $F(z) = 1, z \in \mathbb{B}$. By Propostion 2.9, we have

$$\begin{aligned} \|F\|_*^p &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} (1 - |z|^2)^q G^s(z, a) d\lambda(z) \geq \int_{\mathbb{B}} (1 - |z|^2)^q G^s(z) d\lambda(z) \\ &\simeq \int_{\mathbb{B}} (1 - |z|^2)^q \frac{(1 - |z|^2)^{ns}}{|z|^{2(n-1)s}} d\lambda(z) = \int_{\mathbb{B}} \frac{(1 - |z|^2)^{q+ns-n-1}}{|z|^{2(n-1)s}} dV(z) \\ &= I_1 + I_2, \end{aligned}$$

where

$$I_1 = \int_{\mathbb{B}_{1/2}} \frac{(1 - |z|^2)^{q+ns-n-1}}{|z|^{2(n-1)s}} dV(z)$$

and

$$I_2 = \int_{\mathbb{B} \setminus \mathbb{B}_{1/2}} \frac{(1 - |z|^2)^{q+ns-n-1}}{|z|^{2(n-1)s}} dV(z).$$

We consider two cases.

Case I: $ns + q \leq n$. We have $I_2 = \infty$, which is a contradiction. Indeed,

$$I_2 \simeq \int_{\mathbb{B} \setminus \mathbb{B}_{1/2}} (1 - |z|^2)^{q+ns-n-1} dV(z) \simeq \int_{1/2}^1 \frac{1}{(1 - r)^{n+1-q-ns}} dr = \infty.$$

Case II: $s \geq \frac{n}{n-1}$. We claim in this case, $I_1 = \infty$, which contradicts to the assumption that $F \in \mathcal{N}_*(p, q, s)$. Indeed, we have

$$I_1 \simeq \int_{\mathbb{B}_{1/2}} \frac{1}{|z|^{2(n-1)s}} dV(z) \simeq \int_0^{1/2} \frac{1}{r^{2(n-1)s-2n+1}} dr = \infty.$$

• **Sufficiency.** Suppose $ns + q > n, s < \frac{n}{n-1}$ and P is a polynomial defined on \mathbb{B} . Then we have

$$\begin{aligned} \|P\|_*^p &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |P(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) \\ &\lesssim \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} (1 - |z|^2)^q G^s(z, a) d\lambda(z) \quad (P \text{ is bounded on } \mathbb{B}) \\ &\simeq \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} (1 - |z|^2)^p \frac{(1 - |\Phi_a(z)|^2)^{ns}}{|\Phi_a(z)|^{2(n-1)s}} d\lambda(z) \\ &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} (1 - |\Phi_a(w)|^2)^q \frac{(1 - |w|^2)^{ns}}{|w|^{2(n-1)s}} d\lambda(w) \\ &\quad (\text{change variable } w = \Phi_a(z)) \\ &= \sup_{a \in \mathbb{B}} (1 - |a|^2)^q \int_{\mathbb{B}} \frac{(1 - |w|^2)^{q+ns-n-1}}{|w|^{2(n-1)s} |1 - \langle a, w \rangle|^{2q}} dV(w) \end{aligned}$$

For each $a \in \mathbb{B}$, consider the term

$$\int_{\mathbb{B}} \frac{(1 - |w|^2)^{q+ns-n-1}}{|w|^{2(n-1)s} |1 - \langle a, w \rangle|^{2q}} dV(w) = I_{1,a} + I_{2,a},$$

where

$$I_{1,a} = \int_{\mathbb{B}_{1/2}} \frac{(1 - |w|^2)^{q+ns-n-1}}{|w|^{2(n-1)s} |1 - \langle a, w \rangle|^{2q}} dV(w)$$

and

$$I_{2,a} = \int_{\mathbb{B} \setminus \mathbb{B}_{1/2}} \frac{(1 - |w|^2)^{q+ns-n-1}}{|w|^{2(n-1)s} |1 - \langle a, w \rangle|^{2q}} dV(w)$$

For $I_{1,a}$, by the proof of necessary part, we have

$$I_{1,a} \simeq \int_{\mathbb{B}_{1/2}} \frac{1}{|w|^{2(n-1)s}} dV(w) < M,$$

for some $M > 0$, which is independent of the choice of a . Thus, we have $\sup_{a \in \mathbb{B}} (1 - |a|^2)^q I_{1,a} < \infty$.

For $I_{2,a}$, by [25, Theorem 1.12], we have

$$\begin{aligned} I_{2,a} &\simeq \int_{\mathbb{B} \setminus \mathbb{B}_{1/2}} \frac{(1 - |w|^2)^{q+ns-n-1}}{|w|^{2(n-1)s} |1 - \langle a, w \rangle|^{2q}} dV(w) \\ &\lesssim \int_{\mathbb{B}} \frac{(1 - |w|^2)^{q+ns-n-1}}{|1 - \langle a, w \rangle|^{2q}} dV(w) \\ &\simeq \begin{cases} \text{bounded in } \mathbb{B}, & \text{if } ns > q; \\ \log \frac{1}{1 - |a|^2}, & \text{if } ns = q; \\ (1 - |a|^2)^{ns-q}, & \text{if } ns < q, \end{cases} \end{aligned}$$

which implies $\sup_{a \in \mathbb{B}} (1 - |a|^2)^q I_{2,a} < \infty$.

Combing the above estimations, we see that $\|P\|_* < \infty$, which implies the desired result. \square

The above proposition provide us a hint on describing the $\mathcal{N}(p, q, s)$ -spaces by using the invariant Green's function. More precisely, we have the following result.

Theorem 2.12. *Let $p \geq 1, q, s > 0$ and $n \geq 2$. If $s < \frac{n}{n-1}$, then $\mathcal{N}(p, q, s) = \mathcal{N}_*(p, q, s)$. In particular, if $1 < s < \frac{n}{n-1}$, then $\mathcal{N}(p, q, s) = \mathcal{N}_*(p, q, s) = A^{-\frac{q}{p}}(\mathbb{B})$.*

Proof. Clearly, $\mathcal{N}_*(p, q, s) \subseteq \mathcal{N}(p, q, s)$ and hence it suffices to show $\mathcal{N}(p, q, s) \subseteq \mathcal{N}_*(p, q, s)$ when $s < \frac{n}{n-1}$. Take $f \in \mathcal{N}_*(p, q, s)$. For each $a \in \mathbb{B}$, we have

$$\begin{aligned} & \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) \\ & \simeq \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q \frac{(1 - |\Phi_a(z)|^2)^{ns}}{|\Phi_a(z)|^{2(n-1)s}} d\lambda(z) \\ & = \int_{\mathbb{B}} |f(\Phi_a(w))|^p (1 - |\Phi_a(w)|^2)^q \frac{(1 - |w|^2)^{ns}}{|w|^{2(n-1)s}} d\lambda(w) \\ & \quad (\text{change variable } w = \Phi_a(z)) \\ & = J_{1,a} + J_{2,a}, \end{aligned}$$

where

$$J_{1,a} = \int_{\mathbb{B}_{1/2}} |f(\Phi_a(w))|^p (1 - |\Phi_a(w)|^2)^q \frac{(1 - |w|^2)^{ns}}{|w|^{2(n-1)s}} d\lambda(w)$$

and

$$J_{2,a} = \int_{\mathbb{B} \setminus \mathbb{B}_{1/2}} |f(\Phi_a(w))|^p (1 - |\Phi_a(w)|^2)^q \frac{(1 - |w|^2)^{ns}}{|w|^{2(n-1)s}} d\lambda(w).$$

For $J_{2,a}$, we have

$$\begin{aligned} J_{2,a} & \lesssim \int_{\mathbb{B} \setminus \mathbb{B}_{1/2}} |f(\Phi_a(w))|^p (1 - |\Phi_a(w)|^2)^q (1 - |w|^2)^{ns} d\lambda(w) \\ & \leq \int_{\mathbb{B}} |f(\Phi_a(w))|^p (1 - |\Phi_a(w)|^2)^q (1 - |w|^2)^{ns} d\lambda(w) \\ & = \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \leq \|f\|^p. \end{aligned}$$

For $J_{1,a}$, by Proposition 2.1, $s < \frac{n}{n-1}$ and the proof of Proposition 2.11, we have

$$\begin{aligned} J_{1,a} & \leq |f|_{q/p}^p \int_{\mathbb{B}_{1/2}} \frac{(1 - |w|^2)^{ns-n-1}}{|w|^{2(n-1)s}} dV(w) \\ & \lesssim \|f\|^p \int_{\mathbb{B}_{1/2}} \frac{1}{|w|^{2(n-1)s}} dV(w) \leq M \|f\|^p, \end{aligned}$$

for some $M > 0$.

Hence, combining the estimation on $J_{1,a}$ and $J_{2,a}$, we have, for each $a \in \mathbb{B}$,

$$\int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) \lesssim \|f\|^p,$$

which implies $\|f\|_* \lesssim \|f\|$, that is, $\mathcal{N}(p, q, s) \subseteq \mathcal{N}_*(p, q, s)$ if $s < \frac{n}{n-1}$.

Finally, when $1 < s < \frac{n}{n-1}$, the desired result follows from the above argument and Proposition 2.4. \square

The following result is hence straightforward from Theorem 2.12.

Corollary 2.13. *Let $n \geq 2$ and $0 < s < \frac{n}{n-1}$. Then*

$$\mathcal{N}_{ns} = \mathcal{N}(2, n+1, s) = \mathcal{N}_*(2, n+1, s).$$

Moreover, if $1 < s < \frac{n}{n-1}$, then $A^{-\frac{n+1}{2}} = \mathcal{N}(2, n+1, s) = \mathcal{N}_*(2, n+1, s)$.

Our next result shows that the $\mathcal{N}_*(p, q, s)$ is trivial if $s \geq \frac{n}{n-1}$.

Proposition 2.14. *Let $p \geq 1, q, s > 0$ and $n \geq 2$. If $s \geq \frac{n}{n-1}$, then $\mathcal{N}_*(p, q, s)$ only contains the zero function.*

Proof. We prove it by contradiction. Assume there exists a $a_0 \in \mathbb{B}$, such that $|f(a_0)| \geq \delta > 0$, hence, there exists a $r > 0$, such that

$$|f(\Phi_a(w))| \geq \frac{\delta}{2}, \quad |w| < r.$$

Then we have

$$\begin{aligned} \|f\|_*^p &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q G^s(z, a) d\lambda(z) \\ &\geq \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q G^s(z, a_0) d\lambda(z) \\ &\gtrsim \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q \frac{(1 - |\Phi_{a_0}(z)|^2)^{ns}}{|\Phi_{a_0}(z)|^{2(n-1)s}} d\lambda(z) \\ &= \int_{\mathbb{B}} |f(\Phi_{a_0}(w))|^p (1 - |\Phi_{a_0}(w)|^2)^q \frac{(1 - |w|^2)^{ns-n-1}}{|w|^{2(n-1)s}} dV(w) \\ &\quad (\text{change variable } z = \Phi_a(w)) \\ &\geq \int_{|w| < r} |f(\Phi_{a_0}(w))|^p (1 - |\Phi_{a_0}(w)|^2)^q \frac{(1 - |w|^2)^{ns-n-1}}{|w|^{2(n-1)s}} dV(w) \\ &\geq \left(\frac{\delta}{2}\right)^p (1 - |a_0|^2)^q \int_{|w| < r} \frac{(1 - |w|^2)^{ns+q-n-1}}{|w|^{2(n-1)s} |1 - \langle a, w \rangle|^{2q}} dV(w) \\ &\gtrsim \int_{|w| < r} \frac{1}{|w|^{2(n-1)s}} dV(w) \gtrsim \int_0^r \frac{1}{t^{2(n-1)s-2n+1}} dt = \infty, \end{aligned}$$

which is a contradiction. \square

Remark 2.15. By Theorem 2.12 and Proposition 2.14, it is clear that $\mathcal{N}_*(p, q, s)$ -type space is a special case of $\mathcal{N}(p, q, s)$ -type space. Hence, in the sequel, we will focus our interest on $\mathcal{N}(p, q, s)$ -type space.

3. HADAMARD GAPS IN $\mathcal{N}(p, q, s)$ -TYPE SPACES

A holomorphic function f on \mathbb{B} written in the form

$$f(z) = \sum_{k=0}^{\infty} P_{n_k}(z),$$

where P_{n_k} is a homogeneous polynomial of degree n_k , is said to have *Hadamard gaps* if for some $c > 1$,

$$n_{k+1}/n_k \geq c, \forall k \geq 0.$$

(see, e.g., [19])

Given a Hadamard gap series, we are interested in the following question: for $p \geq 1$ and $q, s > 0$, when does this Hadamard gap series belongs to the $\mathcal{N}(p, q, s)$ spaces?

Observing that a constant function has Hadamard gaps, and hence by Proposition 2.8 or Proposition 2.11, we always assume that the condition $ns + q > n$ holds, that is, $s > 1 - \frac{q}{n}$. Moreover, note that by Proposition 2.4, if $s > 1$, then $\mathcal{N}(p, q, s) = A^{-\frac{q}{p}}(\mathbb{B})$ and for this case, it was already studied by the authors in [3, Theorem 2.5]. Hence, we also assume that $s \leq 1$ in this section.

To formulate our main result in this section, we denote

$$M_k = \sup_{\xi \in \mathbb{S}} |P_{n_k}(\xi)| \quad \text{and} \quad L_{k,p} = \left(\int_{\mathbb{S}} |P_{n_k}(\xi)|^p d\sigma(\xi) \right)^{1/p}, \quad p \geq 1,$$

where $d\sigma$ is the normalized surface measure on \mathbb{S} , that is, $\sigma(\mathbb{S}) = 1$. Clearly for each $k \geq 0$ and $p \geq 1$, M_k and $L_{k,p}$ are well-defined.

We have the following result.

Theorem 3.1. *Let $p \geq 1, q > 0$ and $\max\{0, 1 - \frac{q}{n}\} < s \leq 1$ and $f(z) = \sum_{k=0}^{\infty} P_{n_k}(z)$ with Hadamard gaps. Consider the following statements*

- (a) $\sum_{k=0}^{\infty} \frac{1}{2^{k(ns+q-n)}} \left(\sum_{2^k \leq n_j < 2^{k+1}} M_j^p \right) < \infty;$
- (b) $f \in \mathcal{N}^0(p, q, s);$
- (c) $f \in \mathcal{N}(p, q, s);$
- (d) $\sum_{k=0}^{\infty} \frac{1}{2^{k(ns+q-n)}} \left(\sum_{2^k \leq n_j < 2^{k+1}} L_{j,p}^p \right) < \infty.$

We have $(a) \implies (b) \implies (c) \implies (d)$.

Proof. • $(a) \implies (b)$. Suppose that (a) holds. First, we prove that $f \in \mathcal{N}(p, q, s)$. For $f(z) = \sum_{k=0}^{\infty} P_{n_k}(z)$, using the polar coordinates

and [25, Lemma 1.8], we have

$$\begin{aligned}
\|f\|^p &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left| \sum_{k=0}^{\infty} P_{n_k}(z) \right|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\
&\leq \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left(\sum_{k=0}^{\infty} |P_{n_k}(z)| \right)^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\
&= \sup_{a \in \mathbb{B}} \left\{ \int_{\mathbb{B}} \left(\sum_{k=0}^{\infty} |P_{n_k}(z)| \right)^p \frac{(1 - |a|^2)^{ns} (1 - |z|^2)^{ns+q-n-1}}{|1 - \langle a, z \rangle|^{2ns}} dV(z) \right\} \\
&\lesssim \sup_{a \in \mathbb{B}} \left\{ \int_0^1 (1 - |r|^2)^{ns+q-n-1} \left[\int_{\mathbb{S}} \frac{(1 - |a|^2)^{ns} \left(\sum_{n=0}^{\infty} |P_{n_k}(r\xi)| \right)^p}{|1 - \langle a, r\xi \rangle|^{2ns}} d\sigma(\xi) \right] dr \right\}.
\end{aligned}$$

Since for each $k \in \mathbb{N}$, P_{n_k} is homogeneous, we have for each $\xi \in \mathbb{S}$,

$$(3.1) \quad \sum_{k=0}^{\infty} |P_{n_k}(r\xi)| = \sum_{k=0}^{\infty} |P_{n_k}(\xi) r^{n_k}| \leq \sum_{k=0}^{\infty} M_{n_k} r^{n_k}.$$

Moreover, for each $a \in \mathbb{B}$, by [25, Theorem 1.12], we have

$$\begin{aligned}
&\int_{\mathbb{S}} \frac{1}{|1 - \langle r\xi, a \rangle|^{2ns}} d\sigma(\xi) = \int_{\mathbb{S}} \frac{1}{|1 - \langle \xi, ra \rangle|^{2ns}} d\sigma(\xi) \\
&\simeq \begin{cases} \text{bounded in } \mathbb{B}, & \text{if } s < \frac{1}{2}; \\ \log \frac{1}{1-r^2|a|^2} \leq \log \frac{1}{1-|a|^2}, & \text{if } s = \frac{1}{2}; \\ (1 - r^2|a|^2)^{n-2ns} \leq (1 - |a|^2)^{n-2ns}, & \text{if } \frac{1}{2} < s < 1, \end{cases}
\end{aligned}$$

which implies, there exists a positive constant C such that

$$(3.2) \quad \sup_{a \in \mathbb{B}} (1 - |a|^2)^{ns} \int_{\mathbb{S}} \frac{1}{|1 - \langle r\xi, a \rangle|^{2ns}} dV(z) \leq C.$$

Thus, by (3.1), (3.2) and [14, Theorem 1], we have

$$\begin{aligned}
\|f\|^p &\lesssim \int_0^1 \left(\sum_{k=0}^{\infty} M_{n_k} r^{n_k} \right)^p (1 - |r|^2)^{ns+q-n-1} dr \\
&\simeq \sum_{k=0}^{\infty} \frac{1}{2^{k(ns+q-n)}} \left(\sum_{2^k \leq n_j < 2^{k+1}} M_j \right)^p.
\end{aligned}$$

Since f is in the Hadamard gaps class, there exists a constant $c > 1$ such that $n_{j+1} \leq cn_j$ for all $j \geq 0$. Hence, the maximum number of n_j 's between 2^k and 2^{k+1} is less or equal to $[\log_c 2] + 1$ for $k = 0, 1, 2, \dots$.

Since for every $k \geq 0$, by Hölder inequality,

$$\left(\sum_{2^k \leq n_j < 2^{k+1}} M_j \right)^p \leq ([\log_c 2] + 1)^{p-1} \left(\sum_{2^k \leq n_j < 2^{k+1}} M_j^p \right),$$

Thus, we have

$$(3.3) \quad \|f\|^p \lesssim \sum_{k=0}^{\infty} \frac{1}{2^{k(ns+q-n)}} \left(\sum_{2^k \leq n_j < 2^{k+1}} M_j^p \right) < \infty,$$

which implies $f \in \mathcal{N}(p, q, s)$.

Next, we prove that $f \in \mathcal{N}(p, q, s)$. Put $f_m(z) = \sum_{k=0}^m P_{n_k}(z)$, $m \in \mathbb{N}$, which is bounded in $\overline{\mathbb{B}}$. Thus, by the proof of Theorem 2.6, we know that for each $m \in \mathbb{N}$, $f_m \in \mathcal{N}^0(p, q, s)$. Moreover, by Corollary 2.7, $\mathcal{N}^0(p, q, s)$ is closed and the set of all polynomials is dense in $\mathcal{N}^0(p, q, s)$, and hence it suffices to show that $\|f_m - f\| \rightarrow 0$ as $m \rightarrow \infty$. By (3.3), we have

$$(3.4) \quad \|f_m - f\|^p \lesssim \sum_{k=m'}^{\infty} \left(\frac{1}{2^{k(ns+q-n)}} \sum_{2^k \leq n_j < 2^{k+1}} M_j^p \right),$$

where $m' = \left\lceil \frac{m+1}{[\log_c 2] + 1} \right\rceil$. The result follows from condition (a) and (3.4).

• (b) \implies (c). It is obvious.

• (c) \implies (d). Suppose $f \in \mathcal{N}(p, q, s)$. As the proof in [19, Theorem 1], we have

$$\begin{aligned} \|f\|^p &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left| \sum_{k=0}^{\infty} P_{n_k}(z) \right|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\ &\geq \int_{\mathbb{B}} \left| \sum_{k=0}^{\infty} P_{n_k}(z) \right|^p (1 - |z|^2)^{q+ns-n-1} dV(z) \\ &\simeq \int_{\mathbb{S}} \left(\sum_{k=0}^{\infty} \frac{1}{2^{k(ns+q-n)}} \sum_{2^k \leq n_j < 2^{k+1}} |P_{n_k}(\xi)|^p \right) d\sigma(\xi) \\ &= \sum_{k=0}^{\infty} \frac{1}{2^{k(ns+q-n)}} \left(\sum_{2^k \leq n_j < 2^{k+1}} L_j^p \right), \end{aligned}$$

which implies the desired result. \square

Letting $q = n + 1, p = 2$ and $0 < s < 1$ in Theorem 3.1, we get [3, Theorem 2.1] as a particular case. Generally, condition (d) does not imply condition (a), an example can be found in [3, Remark 2.2].

Next we consider some special cases when all the conditions in Theorem 3.1 are equivalent.

In [23, Corollary 1], for $p > 0$, the authors constructed a sequence of homogeneous polynomials $\{W_k\}_{k \in \mathbb{N}}$ satisfying $\deg(W_k) = k$,

$$\sup_{\xi \in \mathbb{S}} |W_k(\xi)| = 1 \quad \text{and} \quad \int_{\mathbb{S}} |W_k(\xi)|^p d\sigma(\xi) \geq C(p, n)$$

where $C(p, n)$ is a positive constant depending on p and n .

An immediate corollary of Theorem 3.1 is stated as follows.

Corollary 3.2. *Let $p \geq 1, q > 0$ and $\max\{0, 1 - \frac{q}{n}\} < s \leq 1$ and $f(z) = \sum_{k=0}^{\infty} a_k W_{n_k}(z)$ with Hadamard gaps, where $a_k \in \mathbb{C}, k \geq 0$. Then the following statements are equivalent.*

- (a) $\sum_{k=0}^{\infty} \frac{1}{2^{k(ns+q-n)}} \left(\sum_{2^k \leq n_j < 2^{k+1}} |a_j|^p \right) < \infty;$
- (b) $f \in \mathcal{N}^0(p, q, s);$
- (c) $f \in \mathcal{N}(p, q, s).$

Proof. The desired result follows from the fact that for each $k \geq 0, M_k \simeq L_{k,p}$. \square

Let $p \geq 1, q > 0$ and $\max\{0, 1 - \frac{q}{n}\} < s_1 < s_2 \leq 1$. It is clear that

$$(3.5) \quad \mathcal{N}(p, q, s_1) \subseteq \mathcal{N}(p, q, s_2) \subseteq A^{-\frac{q}{p}}(\mathbb{B}).$$

The second application of our main result in this section is to show that the inclusions in (3.5) is strict.

Corollary 3.3. *Let $p \geq 1, q > 0$ and $\max\{0, 1 - \frac{q}{n}\} < s_1 < s_2 \leq 1$. Then we have*

$$\mathcal{N}(p, q, s_1) \subsetneq \mathcal{N}(p, q, s_2) \subsetneq A^{-\frac{q}{p}}(\mathbb{B}).$$

Proof. For the first assertion, consider the series $f_1(z) = \sum_{k=0}^{\infty} 2^{\frac{kq}{p}} W_{2^k}(z)$.

On one hand, by [3, Theorem 2.5], $f_1 \in A^{-\frac{q}{p}}$. On the other hand,

$$\sum_{k=0}^{\infty} \frac{1}{2^{k(ns_2+q-n)}} \left(\sum_{2^k \leq n_j < 2^{k+1}} \left| 2^{\frac{kq}{p}} \right|^p \right) = \sum_{k=0}^{\infty} \frac{1}{2^{k(ns_2-n)}} = \infty,$$

which, by Corollary 3.2, implies $f_1 \notin \mathcal{N}(p, q, s_2)$.

For the second part of the corollary, consider the series $f_2(z) = \sum_{k=0}^{\infty} 2^{\frac{k(ns_1+q-n)}{p}} W_{2^k}(z)$. On one hand, by Corollary 3.2,

$$\sum_{k=0}^{\infty} \frac{1}{2^{k(ns_2+q-n)}} \left(\sum_{2^k \leq n_j < 2^{k+1}} \left| 2^{\frac{k(ns_1+q-n)}{p}} \right|^p \right) = \sum_{k=0}^{\infty} \frac{1}{2^{kn(s_2-s_1)}} < \infty,$$

which implies that $f_2 \in \mathcal{N}(p, q, s_2)$. On the other hand,

$$\sum_{k=0}^{\infty} \frac{1}{2^{k(ns_1+q-n)}} \left(\sum_{2^k \leq n_j < 2^{k+1}} \left| 2^{\frac{k(ns_1+q-n)}{p}} \right|^p \right) = \sum_{k=0}^{\infty} 1 = \infty,$$

which, again by Corollary 3.2, implies $f_2 \notin \mathcal{N}(p, q, s_1)$. \square

Letting $q = n + 1, p = 2$ and $0 < s \leq 1$ in Corollary 3.3, we get the corresponding results in [5] as a particular case.

4. CARLESON MEASURE AND HADAMARD PRODUCTS

4.1. Carleson measure. First we give an equivalent expression of $\mathcal{N}(p, q, s)$ -norm by Carleson measures. Recall (see, e.g., [25]) that for $\xi \in \mathbb{S}$ and $r > 0$, a Carleson tube at ξ is defined as

$$Q_r(\xi) = \{z \in \mathbb{B} : |1 - \langle z, \xi \rangle| < r\}.$$

Moreover, we denote $Q(\xi, r) = \{w \in \mathbb{S} : |1 - \langle w, \xi \rangle| < r\}$.

A positive Borel measure μ in \mathbb{B} is called a p -Carleson measure if there exists a constant $C > 0$ such that

$$\mu(Q_r(\xi)) \leq Cr^p$$

for all $\xi \in \mathbb{S}$ and $r > 0$. Moreover, if

$$\lim_{r \rightarrow 0} \frac{\mu(Q_r(\xi))}{r^p} = 0$$

uniformly for $\xi \in \mathbb{S}$, then μ is called a *vanishing p -Carleson measure*.

The following result describes a relationship between functions in $\mathcal{N}(p, q, s)$ as well as $\mathcal{N}^0(p, q, s)$ and Carleson measures.

Proposition 4.1. *Let $f \in H(\mathbb{B})$ and $p \geq 1, q, s > 0$, and $d\mu_{f,p,q,s}(z) = |f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z)$. The following assertions hold.*

- (1) $f \in \mathcal{N}(p, q, s)$ if and only if $d\mu_{f,p,q,s}$ is a (ns) -Carleson measure;
- (2) $f \in \mathcal{N}^0(p, q, s)$ if and only if $d\mu_{f,p,q,s}$ is a vanishing (ns) -Carleson measure.

Moreover, it holds

$$\begin{aligned}
 \|f\|^p &\simeq \sup_{0 < r < 1, \xi \in \mathbb{S}} \frac{\mu_{f,p,q,s}(Q_r(\xi))}{r^{ns}} \\
 (4.1) \quad &= \sup_{0 < r < 1, \xi \in \mathbb{S}} \frac{1}{r^{ns}} \int_{Q_r(\xi)} |f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z).
 \end{aligned}$$

Proof. (1). Note tht for $f \in \mathcal{N}(p, q, s)$, we can write

$$\begin{aligned}
 \|f\|^p &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)|^p \frac{(1 - |a|^2)^{ns} (1 - |z|^2)^{q+ns-n-1}}{|1 - \langle a, z \rangle|^{2ns}} dV(z) \\
 &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left(\frac{1 - |a|^2}{|1 - \langle a, z \rangle|^2} \right)^{ns} d\mu_{f,p,q,s}(z).
 \end{aligned}$$

Then (1) is obtained, by [27, Theorem 45]. Moreover, the equation (4.1) follows from the same [27, Theorem 45].

(2). This is a consequence of the “little-oh version” of [27, Theorem 45]. \square

4.1.1. *Embedding relationship with weighted Bergman space.* For $k > 0$ and f holomorphic in \mathbb{B} , we denote

$$M_k(r, f) = \left(\int_{\mathbb{S}} |f(r\xi)|^k d\sigma(\xi) \right)^{1/k}, \quad 0 \leq r < 1.$$

By using this expression, we can rewrite the norm of $\|\cdot\|_{k,\rho}$ of the weighted Bergman space A_ρ^k with $k \geq 1$ and $\rho > -1$ as follows:

$$\begin{aligned}
 \|f\|_{k,\rho} &\simeq \left(\int_0^1 r^{2n-1} (1 - r^2)^\rho M_k^k(r, f) dr \right)^{1/k} \\
 &\simeq \left(\int_0^1 r^{2n-1} (1 - r)^\rho M_k^k(r, f) dr \right)^{1/k}.
 \end{aligned}$$

As an application of Proposition 4.1, we establish the following embedding relation between A_ρ^k and $\mathcal{N}(p, q, s)$ with some proper condition on k and ρ . Note that, by Proposition 2.8 and the fact that the set of all polynomials belongs to A_ρ^k , it is natural for us to assume that $ns + q > n$.

We have the following result.

Proposition 4.2. *Let $p \geq 1, q > 0$ and $s > \max\{0, 1 - \frac{q}{n}\}$. Then the following assertions hold.*

(1) *If $0 < s < 1$, then for $\max\{0, q - n\} < \rho < \frac{q+ns-n}{1-s}$, we have*

$$\|f\| \lesssim \|f\|_{\frac{p(n+\rho)}{q}, \rho-1}, \quad \text{that is, } A_{\rho-1}^{\frac{p(n+\rho)}{q}} \subseteq \mathcal{N}(p, q, s);$$

(2) If $s \geq 1$, then for $\rho > \max\{0, q-n\}$, we have $\|f\| \lesssim \|f\|_{\frac{p(n+\rho)}{q}, (\rho-1)}$,

that is, $A_{\rho-1}^{\frac{p(n+\rho)}{q}} \subseteq \mathcal{N}(p, q, s)$.

Proof. The proof for (2) is a simple modification of (1) and hence we omit the proof for (2) here. Suppose $0 < s < 1$. Note that since $\rho < \frac{q+ns-n}{1-s}$, we have

$$(q + ns - n - \rho) \cdot \frac{n + \rho}{n + \rho - q} + \rho > 0.$$

By [25, Corollary 5.24] and Hölder's inequality, for fixed $\xi \in \mathbb{S}$ and $0 < r < 1$, we have

$$\begin{aligned} I_{r,\xi} &= \frac{1}{r^{ns}} \int_{Q_r(\xi)} |f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z) \\ &= \frac{1}{r^{ns}} \int_{Q_r(\xi)} |f(z)|^p (1 - |z|^2)^{q+ns-n-1} dV(z) \\ &\simeq \frac{1}{r^{ns}} \int_{Q_r(\xi)} |f(z)|^p (1 - |z|^2)^{q+ns-n-\rho} dV_{\rho-1}(z) \\ &\leq \frac{1}{r^{ns}} \left(\int_{Q_r(\xi)} |f(z)|^{\frac{p(n+\rho)}{q}} dV_{\rho-1}(z) \right)^{\frac{q}{n+\rho}} \\ &\quad \cdot \left(\int_{Q_r(s)} (1 - |z|^2)^{(q+ns-n-\rho) \cdot \frac{n+\rho}{n+\rho-q}} dV_{\rho-1}(z) \right)^{\frac{n+\rho-q}{n+\rho}} \\ &\lesssim \frac{\|f\|_{\frac{p(n+\rho)}{q}, \rho-1}^p}{r^{ns}} \cdot \left(\int_{Q_r(s)} (1 - |z|^2)^{(q+ns-n-\rho) \cdot \frac{n+\rho}{n+\rho-q} + \rho-1} dV(z) \right)^{\frac{n+\rho-q}{n+\rho}} \\ &\simeq \frac{\|f\|_{\frac{p(n+\rho)}{q}, \rho-1}^p}{r^{ns}} \cdot \left(r^{n+1+(q+ns-n-\rho) \cdot \frac{n+\rho}{n+\rho-q} + \rho-1} \right)^{\frac{n+\rho-q}{n+\rho}} \\ &= \|f\|_{\frac{p(n+\rho)}{q}, \rho-1}^p, \end{aligned}$$

which, by Proposition 4.1, implies

$$\|f\|^p \simeq \sup_{\xi \in \mathbb{S}, 0 < r < 1} I_{r,\xi} \lesssim \|f\|_{\frac{p(n+\rho)}{q}, \rho-1}^p.$$

Hence, we prove the desired result. \square

Corollary 4.3. Let $p \geq 1, q > 0$ and $s > \max\{0, 1 - \frac{q}{n}\}$. If $q > n$, then we have $\|f\| \lesssim \|f\|_{p, q-n-1}$, that is $A_{q-n-1}^p \subseteq \mathcal{N}(p, q, s)$.

Proof. Note that for fixed $\xi \in \mathbb{S}$ and $0 < r < 1$, if $z \in Q_r(\xi)$, then we have

$$(4.2) \quad r > |1 - \langle z, \xi \rangle| \geq 1 - |\langle z, \xi \rangle| \geq 1 - |z||\xi| = 1 - |z|.$$

For fixed $\xi \in \mathbb{S}$ and $0 < r < 1$, by (4.2), we have

$$\begin{aligned}
I_{r,\xi} &= \frac{1}{r^{ns}} \int_{Q_r(\xi)} |f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z) \\
&= \frac{1}{r^{ns}} \int_{Q_r(\xi)} |f(z)|^p (1 - |z|^2)^{q+ns-n-1} dV(z) \\
&\simeq \frac{1}{r^{ns}} \int_{Q_r(\xi)} |f(z)|^p (1 - |z|^2)^{ns} dV_{q-n-1}(z) \\
&\leq \frac{1}{r^{ns}} \cdot \sup_{z \in Q_r(\xi)} (1 - |z|^2)^{ns} \cdot \int_{Q_r(\xi)} |f(z)|^p dV_{q-n-1}(z) \\
&\lesssim \frac{1}{r^{ns}} \cdot \sup_{z \in Q_r(\xi)} (1 - |z|^2)^{ns} \cdot \|f\|_{p,q-n-1}^p \\
&\leq \|f\|_{p,q-n-1}^p,
\end{aligned}$$

which implies that $A_{q-n-1}^p \subseteq \mathcal{N}(p, q, s)$. \square

Remark 4.4. In Proposition 4.2, in particular, putting $\rho = q+ns-n$, we get $A_{q+ns-n-1}^{\frac{p(q+ns)}{q}} \subseteq \mathcal{N}(p, q, s)$. Moreover, it is clear that $\|\cdot\|_{p,q+ns-n-1} \leq \|\cdot\|$, that is, $\mathcal{N}(p, q, s) \subset A_{q+ns-n-1}^p$. Combining this fact with Proposition 4.2, we have, for $p \geq 1, q > 0$ and $s > \max\{0, 1 - \frac{q}{n}\}$, the following embedding relation holds

$$A_{q+ns-n-1}^{\frac{p(q+ns)}{q}} \subseteq \mathcal{N}(p, q, s) \subseteq A_{q+ns-n-1}^p.$$

4.1.2. Embedding relationship with weighted Hardy space. In this section, by using Proposition 4.1, we will show that for some α and β , the Hardy space $H_\beta^\alpha(\mathbb{B}) \subseteq \mathcal{N}(p, q, s)$ for $p \geq 1, q > 0$ and $s > \max\{0, 1 - \frac{q}{n}\}$. Recall that for $\alpha > 0$, the *Hardy space* H^α consists of holomorphic functions f in \mathbb{B} such that

$$\|f\|_{H^\alpha} = \sup_{0 < r < 1} M_\alpha(r, f) < \infty$$

(see, e.g., [25]). It is well-known that when $1 \leq \alpha < \infty$, H^α is a Banach space with the norm $\|\cdot\|_{H^\alpha}$; if $0 < \alpha < 1$, H^α is a complete metric space.

More generally, the *weighted Hardy space* H_β^α is defined as follows, for $\alpha > 0$ and $\beta \geq 0$,

$$H_\beta^\alpha = \left\{ f \in H(\mathbb{B}) : \|f\|_{H_\beta^\alpha} = \sup_{0 < r < 1} (1 - r)^\beta M_\alpha(r, f) < \infty \right\}.$$

It is known that when $\alpha \geq 1$, H_β^α is a Banach space with the norm $\|\cdot\|_{H_\beta^\alpha}$ (see, e.g., [20]). Moreover, the little weighted Hardy space $H_{\beta,0}^\alpha$ is defined as follows:

$$H_{\beta,0}^\alpha = \left\{ f \in H(\mathbb{B}) : \lim_{r \rightarrow 1^-} (1-r)^\beta M_\alpha(r, f) = 0 \right\}.$$

It is easy to see that $H_{\beta,0}^\alpha \subseteq H_\beta^\alpha$.

As an application of Proposition 4.1, we have the following result.

Proposition 4.5. *Let $p \geq 1, q > 0$ and $s > \max\{0, 1 - \frac{q}{n}\}$. The following statements hold.*

- (1) *If $0 < s < 1$, then $H_{\frac{q}{p} - \frac{n}{\alpha}}^\alpha \subseteq \mathcal{N}(p, q, s)$, where $\max\left\{p, \frac{np}{q}\right\} \leq \alpha < \frac{p}{1-s}$;*
- (2) *If $s \geq 1$, then $H_{\frac{q}{p} - \frac{n}{\alpha}}^\alpha \subseteq \mathcal{N}(p, q, s)$, where $\alpha \geq \max\left\{p, \frac{np}{q}\right\}$. In particular, if $\alpha \geq \max\left\{p, \frac{np}{q}\right\}$, then $H_{\frac{q}{p} - \frac{n}{\alpha}}^\alpha \subseteq A^{-\frac{q}{p}}$.*

Proof. The proof for (2) is a simple modification of (1) and hence we omit the proof for (2) here.

Note that for fixed $\xi \in \mathbb{S}$ and $0 < r < 1$, if $z \in Q_r(\xi)$, by the argument in Corollary 4.3, we have

$$(4.3) \quad 1 - r < |z| < 1.$$

We consider two different cases.

Case I: $\alpha = p$. First we note that by condition, $p = \alpha \geq \frac{np}{q}$, which implies $q \geq n$. For each $\xi \in \mathbb{S}$ and $0 < r < 1$, by (4.3), we have

$$\begin{aligned} I_{r,\xi} &= \frac{1}{r^{ns}} \int_{Q_r(\xi)} |f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z) \\ &= \frac{1}{r^{ns}} \int_{\{z \in \mathbb{B} : |1 - \langle z, \xi \rangle| < r\}} |f(z)|^p (1 - |z|^2)^{q+ns-n-1} dV(z) \\ &\lesssim \frac{1}{r^{ns}} \int_{1-r}^1 (1 - t^2)^{q+ns-n-1} \left(\int_{Q(\xi, r)} |f(\gamma t)|^p d\sigma(\gamma) \right) dt \\ &= \frac{1}{r^{ns}} \int_{1-r}^1 \frac{(1 - t^2)^{q+ns-n-1}}{(1 - t)^{q-n}} (1 - t)^{q-n} M_p^p(t, f) dt \\ &\leq \|f\|_{H_{\frac{q-n}{p}}^p}^p \cdot \frac{1}{r^{ns}} \int_{1-r}^1 (1 - t)^{ns-1} dt \simeq \|f\|_{H_{\frac{q-n}{p}}^p}^p. \end{aligned}$$

Thus, we have

$$\|f\|^p = \sup_{0 < r < 1, \xi \in \mathbb{S}} I_{r,\xi} \lesssim \|f\|_{H_{\frac{q-n}{p}}^p}^p,$$

which implies the desired result.

Case II: $\alpha > p$. For each $\xi \in \mathbb{S}$ and $0 < r < 1$, by Hölder inequality and [25, Lemma 4.6], we have

$$\begin{aligned} \int_{Q(\xi, r)} |f(\gamma t)|^p d\sigma(\gamma) &\leq \sigma(Q(\xi, r))^{1-\frac{p}{\alpha}} \left(\int_{Q(\xi, r)} |f(\gamma t)|^\alpha d\sigma(\gamma) \right)^{\frac{p}{\alpha}} \\ &\lesssim \frac{r^{n-\frac{np}{\alpha}} (1-t)^{q-\frac{np}{\alpha}} M_\alpha^p(t, f)}{(1-t)^{q-\frac{np}{\alpha}}} \\ &\leq \frac{r^{n-\frac{np}{\alpha}} \|f\|_{H_{\frac{q}{p}-\frac{n}{\alpha}}^\alpha}^p}{(1-t)^{q-\frac{np}{\alpha}}}. \end{aligned}$$

Then, by (4.3) and previous calculation, we have

$$\begin{aligned} I_{r, \xi} &\lesssim \frac{1}{r^{ns}} \int_{1-r}^1 (1-t^2)^{q+ns-n-1} \left(\int_{Q(\xi, r)} |f(\gamma t)|^p d\sigma(\gamma) \right) dt \\ &\lesssim r^{n-\frac{np}{\alpha}-ns} \|f\|_{H_{\frac{q}{p}-\frac{n}{\alpha}}^\alpha}^p \int_{1-r}^1 \frac{(1-t)^{q+ns-n-1}}{(1-t)^{q-\frac{np}{\alpha}}} dt \\ &= r^{n-\frac{np}{\alpha}-ns} \|f\|_{H_{\frac{q}{p}-\frac{n}{\alpha}}^\alpha}^p \int_{1-r}^1 (1-t)^{ns+\frac{np}{\alpha}-n-1} dt \\ &\simeq \|f\|_{H_{\frac{q}{p}-\frac{n}{\alpha}}^\alpha}^p. \end{aligned}$$

Thus,

$$\|f\|^p \simeq \sup_{0 < r < 1, \xi \in \mathbb{S}} I_{r, \xi} \lesssim \|f\|_{H_{\frac{q}{p}-\frac{n}{\alpha}}^\alpha}^p,$$

and hence the desired result follows. \square

The following result describes the behavior of Hadamard gap series in H_β^α , whose idea comes from [11, Theorem 1].

Proposition 4.6. *Let $\alpha > 0$, $\beta > 0$ and $f(z) = \sum_{k=1}^\infty P_{n_k}(z)$ with Hadamard gaps. Then the following statements hold true:*

- (1) $f \in H_\beta^\alpha$ if and only if $\sup_{k \geq 1} \frac{L_{k, \alpha}}{n_k^\beta} < \infty$;
- (2) $f \in H_{\beta, 0}^\alpha$ if and only if $\lim_{k \rightarrow \infty} \frac{L_{k, \alpha}}{n_k^\beta} = 0$,

where $L_{k, \alpha} = \left(\int_{\mathbb{S}} |P_{n_k}(\xi)|^\alpha d\sigma(\xi) \right)^{\frac{1}{\alpha}}$.

Proof. (1). **Necessity.** Let $f \in H_\beta^\alpha$. By [18, Proposition 1.4.7] and [11, Lemma 1], we have, for each $k \in \mathbb{N}$,

$$\begin{aligned}
 \int_{\mathbb{S}} |f(r\xi)|^\alpha d\sigma(\xi) &= \int_{\mathbb{S}} \left(\int_0^{2\pi} |f(r\xi e^{i\theta})|^\alpha \frac{d\theta}{2\pi} \right) d\sigma(\xi) \\
 &= \int_{\mathbb{S}} \left(\int_0^{2\pi} \left| \sum_{k=0}^{\infty} P_{n_k}(r\xi e^{i\theta}) \right|^\alpha \frac{d\theta}{2\pi} \right) d\sigma(\xi) \\
 (4.4) \quad &\simeq \int_{\mathbb{S}} \left(\sum_{k=0}^{\infty} |P_{n_k}(\xi)|^2 r^{2n_k} \right)^{\alpha/2} d\sigma(\xi) \\
 &\geq r^{\alpha n_k} \int_{\mathbb{S}} |P_{n_k}(\xi)|^\alpha d\sigma(\xi).
 \end{aligned}$$

Hence,

$$(4.5) \quad (1-r)^\beta r^{n_k} L_{k,\alpha} \leq (1-r)^\beta M_\alpha(r, f) \leq \|f\|_{H_\beta^\alpha}.$$

Choosing $r = 1 - \frac{1}{n_k}$ and using the well-known inequality $(1 + \frac{1}{m})^{m+1} \leq 4$, $m \in \mathbb{N}$, we obtain

$$\sup_{k \in \mathbb{N}} \frac{L_{k,\alpha}}{n_k^\beta} \leq C \|f\|_{H_\beta^\alpha},$$

as desired.

Sufficiency. Suppose that $\sup_{k \in \mathbb{N}} \frac{L_{k,\alpha}}{n_k^\beta} < \infty$. For a fixed $r \in (0, 1)$, we have

$$\frac{\sum_{k=0}^{\infty} r^{\alpha n_k} n_k^{\alpha\beta}}{1 - r^\alpha} = \left(\sum_{k=0}^{\infty} r^{\alpha n_k} n_k^{\alpha\beta} \right) \cdot \left(\sum_{s=0}^{\infty} r^{\alpha s} \right) = \sum_{t=0}^{\infty} \left(\sum_{n_j \leq t} n_j^{\alpha\beta} \right) r^{\alpha t}.$$

Since,

$$\lim_{k \rightarrow \infty} \frac{k^{\alpha\beta} k!}{(\alpha\beta)(\alpha\beta+1) \dots (\alpha\beta+k)} = \Gamma(\alpha\beta), \alpha\beta > 0,$$

we have

$$\sup_{k \in \mathbb{N}} \left(\frac{k^{\alpha\beta} k!}{(k + \alpha\beta)(k + \alpha\beta - 1) \dots (\alpha\beta + 1)} \right) \leq M,$$

where M is some positive number depending on α and β . Hence, for each $k \geq 0$,

$$\begin{aligned}
 \frac{k^{\alpha\beta}}{(-1)^k \binom{-\alpha\beta-1}{k}} &= \frac{k^{\alpha\beta} k!}{(-1)^k (-\alpha\beta-1)(-\alpha\beta-2) \dots (-\alpha\beta-k)} \\
 (4.6) \quad &= \frac{k^{\alpha\beta} k!}{(k + \alpha\beta)(k + \alpha\beta - 1) \dots (\alpha\beta + 1)} \leq M,
 \end{aligned}$$

where $\binom{\gamma}{k} = \frac{\gamma(\gamma-1)\dots(\gamma-k+1)}{k!}$, $\gamma \in \mathbb{R}$.

Moreover, since f is in Hadamard gaps class, there exists a constant $c > 1$ such that $n_{j+1} \geq cn_j$ for all $j \geq 0$. Hence

$$(4.7) \quad \frac{1}{t^{\alpha\beta}} \left(\sum_{n_j \leq t} n_j^{\alpha\beta} \right) \leq \sum_{m=0}^{\infty} \left(\frac{1}{c^{\alpha\beta}} \right)^m = \frac{c^{\alpha\beta}}{c^{\alpha\beta} - 1}.$$

Combining (4.6) and (4.7), we have

$$\frac{t^{\alpha\beta}}{(-1)^t \binom{-\alpha\beta-1}{t}} \cdot \frac{1}{t^{\alpha\beta}} \left(\sum_{n_j \leq t} n_j^{\alpha\beta} \right) \leq \frac{Mc^{\alpha\beta}}{c^{\alpha\beta} - 1},$$

which implies

$$(4.8) \quad \sum_{n_j \leq t} n_j^{\alpha\beta} \leq (-1)^t \binom{-\alpha\beta-1}{t} \frac{Mc^{\alpha\beta}}{c^{\alpha\beta} - 1}.$$

Hence, by (4.8), we have

$$\frac{\sum_{k=0}^{\infty} r^{\alpha n_k} n_k^{\alpha\beta}}{1 - r^{\alpha}} \lesssim \sum_{r=0}^{\infty} (-1)^t \binom{-\alpha\beta-1}{t} r^{\alpha t} = \frac{1}{(1 - r^{\alpha})^{\alpha\beta+1}},$$

which implies for $\alpha, \beta > 0$,

$$(4.9) \quad (1 - r^{\alpha})^{\alpha\beta} \sum_{k=0}^{\infty} r^{\alpha n_k} n_k^{\alpha\beta} \lesssim 1.$$

Now we consider two different cases.

Case I: $\alpha \in (0, 2]$. From (4.4), (4.9) and by using the known inequality

$$\left(\sum_{k=1}^{\infty} a_k \right)^q \leq \sum_{k=1}^{\infty} a_k^q,$$

where $a_k \geq 0, k \in \mathbb{N}, q \in [0, 1]$, we have that

$$\begin{aligned}
 \|f\|_{H_\beta^\alpha}^\alpha &\simeq \sup_{0 < r < 1} \left[(1-r)^{\alpha\beta} \int_{\mathbb{S}} \left(\sum_{k=0}^{\infty} |P_{n_k}(\xi)|^2 r^{2n_k} \right)^{\alpha/2} d\sigma(\xi) \right] \\
 &\leq \sup_{0 < r < 1} \left[(1-r)^{\alpha\beta} \int_{\mathbb{S}} \left(\sum_{k=0}^{\infty} |P_{n_k}(\xi)|^\alpha r^{\alpha n_k} \right) d\sigma(\xi) \right] \\
 &= \sup_{0 < r < 1} \left[(1-r)^{\alpha\beta} \cdot \sum_{k=0}^{\infty} r^{\alpha n_k} L_{k,\alpha}^\alpha \right] \\
 &\lesssim \sup_{0 < r < 1} \left[(1-r)^{\alpha\beta} \cdot \sum_{k=0}^{\infty} r^{\alpha n_k} n_k^{\alpha\beta} \right] \\
 &\lesssim \sup_{0 < r < 1} \left(\frac{1-r}{1-r^\alpha} \right)^{\alpha\beta} < \infty,
 \end{aligned}$$

which implies the desired result.

Case II: $\alpha > 2$. For each $r \in (0, 1)$, by Minkowski's inequality and (4.9), we have

$$\begin{aligned}
 \left[\int_{\mathbb{S}} \left(\sum_{k=0}^{\infty} |P_{n_k}(\xi)|^2 r^{2n_k} \right)^{\alpha/2} d\sigma(\xi) \right]^{\frac{2}{\alpha}} &\leq \sum_{k=0}^{\infty} \left(\int_{\mathbb{S}} (|P_{n_k}(\xi)|^2 r^{2n_k})^{\frac{\alpha}{2}} d\sigma(\xi) \right)^{\frac{2}{\alpha}} \\
 (4.10) \qquad \qquad \qquad &= \sum_{k=0}^{\infty} r^{2n_k} L_{k,\alpha}^2 \lesssim \sum_{k=0}^{\infty} r^{2n_k} n_k^{2\beta}.
 \end{aligned}$$

Thus, by (4.4) and (4.9),

$$\begin{aligned}
 \|f\|_{H_\beta^\alpha}^\alpha &\simeq \sup_{0 < r < 1} \left[(1-r)^{\alpha\beta} \int_{\mathbb{S}} \left(\sum_{k=0}^{\infty} |P_{n_k}(\xi)|^2 r^{2n_k} \right)^{\alpha/2} d\sigma(\xi) \right] \\
 (4.11) \qquad &\leq \sup_{0 < r < 1} \left[(1-r)^{\alpha\beta} \left(\sum_{k=0}^{\infty} r^{2n_k} L_{k,\alpha}^2 \right)^{\frac{\alpha}{2}} \right] \\
 &\lesssim \sup_{0 < r < 1} \left[(1-r)^{\alpha\beta} \left(\sum_{k=0}^{\infty} r^{2n_k} n_k^{2\beta} \right)^{\frac{\alpha}{2}} \right] \\
 &\lesssim \sup_{0 < r < 1} \left(\frac{1-r}{1-r^2} \right)^{\alpha\beta} < \infty,
 \end{aligned}$$

and hence $f \in H_\beta^\alpha$.

(2). **Necessity.** Let $f \in H_{\beta,0}^\alpha$. Then for every $\varepsilon > 0$, there is a $\delta > 0$, such that

$$(4.12) \quad (1-r)^\beta M_\alpha(r, f) < \varepsilon$$

whenever $\delta < r < 1$. From the first inequality in (4.5) and (4.12), we have

$$(1-r)^\beta r^{n_k} L_{k,\alpha} < \varepsilon$$

for each $k \in \mathbb{N}$ and $r \in (\delta, 1)$. Choosing $r = 1 - \frac{1}{n_k}$, where $n_k > \frac{1}{1-\delta}$, we obtain

$$\frac{L_{k,\alpha}}{n_k^\beta} < 4\varepsilon.$$

From this and since ε is an arbitrary positive number, it follows that

$$\lim_{k \rightarrow \infty} \frac{L_{k,\alpha}}{n_k^\beta} = 0.$$

Sufficiency. Suppose $\lim_{k \rightarrow \infty} \frac{L_{k,\alpha}}{n_k^\beta} = 0$. Take and fix a $\varepsilon > 0$, there is a $k_0 \in \mathbb{N}$ such that

$$L_{k,\alpha} < \varepsilon^{\frac{1}{\alpha}} n_k^\beta, \quad \text{for } k \geq k_0.$$

Fix the k_0 choosen above. Then, there exists a $\delta > 0$, such that when $\delta < r < 1$,

$$(1-r)^{\alpha\beta} \sum_{k=0}^{k_0} L_{k,\alpha}^\alpha < \varepsilon.$$

Again, we consider two cases as the proof in part (1).

Case I: $\alpha \in (0, 2]$. By (4.5) and the proof in part (1), for $r \in (\delta, 1)$, we have

$$\begin{aligned} (1-r)^{\alpha\beta} M_\alpha^\alpha(r, f) &\simeq (1-r)^{\alpha\beta} \int_{\mathbb{S}} \left(\sum_{k=0}^{\infty} |P_{n_k}(\xi)|^2 r^{2n_k} \right)^{\alpha/2} d\sigma(\xi) \\ &\leq (1-r)^{\alpha\beta} \cdot \left[\left(\sum_{k=0}^{k_0} + \sum_{k=k_0+1}^{\infty} \right) r^{\alpha n_k} L_{k,\alpha}^\alpha \right] \\ &\leq \varepsilon + (1-r)^{\alpha\beta} \cdot \sum_{k=k_0+1}^{\infty} r^{\alpha n_k} L_{k,\alpha}^\alpha \\ &\leq \varepsilon + \varepsilon (1-r)^{\alpha\beta} \cdot \sum_{k=k_0+1}^{\infty} r^{\alpha n_k} n_k^{\alpha\beta} \\ &\lesssim \varepsilon \cdot \left(1 + \sup_{0 < r < 1} \left(\frac{1-r}{1-r^2} \right)^{\alpha\beta} \right) \lesssim \varepsilon, \end{aligned}$$

which implies

$$\lim_{r \rightarrow 1^-} (1-r)^\beta M_\alpha(r, f) = 0$$

and hence $f \in H_{\beta,0}^\alpha$ as desired.

Case II: $\alpha > 2$. The implication for case $p \geq 2$ follows similarly, from (4.10), (4.11) and the known inequality

$$(a+b)^{\frac{\alpha}{2}} \leq 2^{\frac{\alpha}{2}-1} \left(a^{\frac{\alpha}{2}} + b^{\frac{\alpha}{2}} \right), \quad a, b \geq 0.$$

Hence, we omit the detail here. \square

As a corollary of Proposition 4.6, we can show that when $\frac{q}{p} - \frac{n}{\alpha} > 0$, the inclusion in Proposition 4.5 is strict.

Corollary 4.7. *Let p, q, s and α satisfy the condition in Proposition 4.5. If*

$$\frac{q}{p} - \frac{n}{\alpha} > 0,$$

then the inclusion in Proposition 4.5 is strict.

Proof. First we take a sequence of homogeneous polynomials $\{W_k\}_{k \in \mathbb{N}}$ satisfying $\deg(W_k) = k$,

$$\sup_{\xi \in \mathbb{S}} |W_k(\xi)| = 1, \quad \text{and} \quad \int_{\mathbb{S}} |W_k(\xi)|^p d\sigma(\xi) \geq C(p, n) > 0.$$

Note that since $p \leq \alpha$, by Hölder's inequality, we have

$$\int_{\mathbb{S}} |W_k(\xi)|^p d\sigma(\xi) \leq \left(\int_{\mathbb{S}} |W_k(\xi)|^\alpha d\sigma(\xi) \right)^{\frac{p}{\alpha}},$$

that is

$$(4.13) \quad \int_{\mathbb{S}} |W_k(\xi)|^\alpha d\sigma(\xi) \geq \left(\int_{\mathbb{S}} |W_k(\xi)|^p d\sigma(\xi) \right)^{\frac{\alpha}{p}} \geq C^{\frac{\alpha}{p}}(p, n).$$

Note that when $s > 1$, $\mathcal{N}(p, q, s) = A^{-\frac{q}{p}}$, and hence we consider three cases.

Case I: $s > 1$. Consider the series $f_1(z) = \sum_{k=0}^{\infty} 2^{\frac{kq}{p}} W_{2^k}(z)$, which by [3, Theorem 2.5], belongs to $A^{-\frac{q}{p}}$. On the other hand, for each $k \in \mathbb{N}$, by (4.13),

$$\frac{L_{k,\alpha}}{2^{k(\frac{q}{p} - \frac{n}{\alpha})}} = \frac{\left(\int_{\mathbb{S}} \left| 2^{\frac{kq}{p}} W_{2^k}(z) \right|^\alpha d\sigma(\xi) \right)^{\frac{1}{\alpha}}}{2^{k(\frac{q}{p} - \frac{n}{\alpha})}} \geq C^{\frac{1}{p}}(p, n) 2^{\frac{kn}{\alpha}}.$$

Hence, we have

$$\sup_{k \geq 0} \frac{L_{k,\alpha}}{2^{k(\frac{q}{p} - \frac{n}{\alpha})}} = \infty,$$

which implies that $f_1 \notin H_{\frac{q}{p}-\frac{n}{\alpha}}^\alpha$.

Case II: $s = 1$. Take any $\varepsilon \in (0, \frac{np}{\alpha})$ and consider the series $f_2(z) = \sum_{k=0}^{\infty} 2^{\frac{k(q-\varepsilon)}{p}} W_{2^k}(z)$. On one hand, we have

$$\sum_{k=0}^{\infty} \frac{1}{2^{kq}} \cdot \left| 2^{\frac{k(q-\varepsilon)}{p}} \right|^p = \sum_{k=0}^{\infty} \frac{1}{2^{k\varepsilon}} < \infty,$$

which by Corollary 3.2, implies that $f_2 \in \mathcal{N}(p, q, 1)$. However, for each $k \in \mathbb{N}$, by (4.13),

$$\frac{L_{k,\alpha}}{2^{k(\frac{q}{p}-\frac{n}{\alpha})}} = \frac{\left(\int_{\mathbb{S}} \left| 2^{\frac{k(q-\varepsilon)}{p}} W_{2^k}(z) \right|^\alpha d\sigma(\xi) \right)^{\frac{1}{\alpha}}}{2^{k(\frac{q}{p}-\frac{n}{\alpha})}} \geq C^{\frac{1}{p}}(p, n) 2^{k(\frac{n}{\alpha}-\frac{\varepsilon}{p})}.$$

Hence, we have

$$\sup_{k \geq 0} \frac{L_{k,\alpha}}{2^{k(\frac{q}{p}-\frac{n}{\alpha})}} = \infty,$$

which implies that $f_2 \notin H_{\frac{q}{p}-\frac{n}{\alpha}}^\alpha$.

Case III: $0 < s < 1$. Since $\alpha < \frac{p}{1-s}$, we have $\frac{s-1}{p} + \frac{1}{\alpha} > 0$. Take any $\varepsilon \in (0, n(s-1) + \frac{np}{\alpha})$ and consider the series $f_3(z) = \sum_{k=0}^{\infty} 2^{\frac{k(ns+q-n-\varepsilon)}{p}} W_{2^k}(z)$.

On one hand, we have

$$\sum_{k=0}^{\infty} \frac{1}{2^{k(ns+q-n)}} \left| 2^{\frac{k(ns+q-n-\varepsilon)}{p}} \right|^p = \sum_{k=0}^{\infty} \frac{1}{2^{k\varepsilon}} < \infty,$$

which by Corollary 3.2, implies that $f_3 \in \mathcal{N}(p, q, s)$. However, for each $k \in \mathbb{N}$, by (4.13),

$$\frac{L_{k,\alpha}}{2^{k(\frac{q}{p}-\frac{n}{\alpha})}} = \frac{\left(\int_{\mathbb{S}} \left| 2^{\frac{k(ns+q-n-\varepsilon)}{p}} W_{2^k}(z) \right|^\alpha d\sigma(\xi) \right)^{\frac{1}{\alpha}}}{2^{k(\frac{q}{p}-\frac{n}{\alpha})}} \geq C^{\frac{1}{p}}(p, n) 2^{k(\frac{n(s-1)}{p} + \frac{n}{\alpha} - \frac{\varepsilon}{p})}.$$

Hence, we have

$$\sup_{k \geq 0} \frac{L_{k,\alpha}}{2^{k(\frac{q}{p}-\frac{n}{\alpha})}} = \infty,$$

which implies that $f_3 \notin H_{\frac{q}{p}-\frac{n}{\alpha}}^\alpha$. □

4.2. Hadamard products. An important application of those Carleson property of $\mathcal{N}(p, q, s)$ -type spaces is to study Hadamard product in them.

We first set up some basic notations which shall be used in this subsection. For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{Z}_+^n$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$, we let

$$\begin{aligned}\bar{z} &= (\bar{z}_1, \dots, \bar{z}_n), \quad z^\eta = z_1^{\eta_1} \cdots z_n^{\eta_n}, \\ |\eta| &= \eta_1 + \cdots + \eta_n, \quad \eta! = \eta_1! \cdots \eta_n!, \\ \partial_j &= \frac{\partial}{\partial z_j}, \quad 1 \leq j \leq n, \quad \partial^\eta = \partial_1^{\eta_1} \cdots \partial_n^{\eta_n}, \\ \nabla &= (\partial_1, \dots, \partial_n), \quad z \cdot \xi = (z_1 \xi_1, \dots, z_n \xi_n).\end{aligned}$$

The Bloch space on the unit ball, denoted by \mathcal{B} , is defined as the space of $H(\mathbb{B})$ for which

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{B}} (1 - |z|^2) |\nabla f(z)| < \infty.$$

Since \mathbb{B} is a complete Reinhardt domain in \mathbb{C}^n , i.e. $z \in \mathbb{B}$ implies $z \cdot \xi \in \mathbb{B}$ for every $\xi \in \bar{U}$, where U is the unit polydisc in \mathbb{C}^n . Then any $f \in H(\mathbb{B})$ has a unique power series

$$f(z) = \sum_{\eta} a_{\eta} z^{\eta}, \quad z \in \mathbb{B},$$

with $a_{\eta} = \frac{\partial^{\eta} f(0)}{\eta!}$, $\eta \in \mathbb{Z}_+^n$. So $H(\mathbb{B})$ may be regarded as a space of multi-index sequence $\{a_{\eta}\}$.

For $f(z) = \sum_{\eta} a_{\eta} z^{\eta}$, $g(z) = \sum_{\eta} b_{\eta} z^{\eta} \in H(\mathbb{B})$ and $d > 0$, the d -Hadamard product of f and g is defined as follows:

$$(f * g)_d(z) = \sum_{\eta} \omega_{\eta}(d) a_{\eta} b_{\eta} z^{\eta},$$

where

$$\omega_{\eta}(d) = \frac{\eta! \Gamma(n + d)}{\Gamma(n + d + |\eta|)}, \quad \eta \in \mathbb{Z}_+^n.$$

(see, e.g., [1, 12]).

Let us denote $H(U)$ the collection of all holomorphic functions in U and $H^{\infty}(U)$ the Banach space consisting of all bounded holomorphic functions in U . The interesting feature of this product is that it is lying not only in $H(\mathbb{B})$ but also in $H(U)$ and for $f, g \in H(\mathbb{B})$ and any $0 \leq r < 1$,

$$(4.14) \quad (f * g)_d(rz) = \langle f_r, g_{\bar{z}}^* \rangle_d := \int_{\mathbb{B}} f(rw) g(z \cdot \bar{w}) dV_{d-1}(w),$$

for any $z \in U$, where

$$f_r(z) = f(rz), \quad g^*(z) = \overline{g(\bar{z})} \quad \text{and} \quad f_\xi(z) = f(\xi \cdot z), \quad \xi \in \mathbb{B},$$

(see, e.g., [1, Proposition 3.2]).

The following lemma proved in [1, Proposition 3.3] plays an important role in this subsection.

Lemma 4.8. *Let $d > 0$, $1 \leq p_1, p_2, p_3 \leq \infty$ with $1 + \frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$, and let $f, g \in H(\mathbb{B})$. Then $(f * g)_d \in H(U)$ with*

$$\|(f * g)_d\|_{p_3, d-1} \leq \|f\|_{p_1, d-1} \|g\|_{p_2, d-1}.$$

*Moreover, if also $p_3 = \infty$, i.e. $p_2 = p'_1 = p_1/(p_1 - 1)$, then $\|(f * g)_d\|_\infty \leq \|f\|_{p_1, d-1} \|g\|_{p'_1, d-1}$. In particular, $(f * g)_d \in H^\infty(U)$ if $f \in A_{d-1}^{p_1}$ and $g \in A_{d-1}^{p'_1}$.*

Proposition 4.9. *Let $p, r \geq 1, q, d > 0$ and $s > \max\{0, 1 - \frac{q}{n}\}$. Then*

- (1) $(f * g)_d \in \mathcal{N}(p, q, s)$ if $f \in A_{d-1}^r$ and $g \in A_{d-1}^{r'}$, where $r > 1$ and $r' = \frac{r}{r-1}$;
- (2) $(f * g)_d \in \mathcal{N}(p, q, s)$ if $f \in A_{d-1}^1$ and $g \in \mathcal{B}$.

Proof. (1). Since $s > 1 - \frac{q}{n}$, by Proposition 2.8 and Lemma 4.8, we have

$$\|(f * g)_d\| \lesssim \|(f * g)_d\|_\infty \leq \|f\|_{r, d-1} \|g\|_{r', d-1} < \infty,$$

which implies the desired result.

(2). By the proof of [12, Theorem 1], we have

$$|(f * g)_d(z)| \lesssim \|g\|_{\mathcal{B}} \|f\|_{1, d-1},$$

which, again, by the condition $s > 1 - \frac{q}{n}$, implies

$$\|(f * g)_d\| \lesssim \|(f * g)_d\|_\infty \lesssim \|g\|_{\mathcal{B}} \|f\|_{1, d-1}.$$

Hence the desired result follows. \square

By using the embedding relation between $\mathcal{N}(p, q, s)$ and A_ρ^k , we have the following result.

Proposition 4.10. *Let p, q, s, ρ satisfy the conditions in Proposition 4.2. Then for $r_1, r_2 \geq 1$ with satisfying*

$$1 + \frac{q}{p(n + \rho)} = \frac{1}{r_1} + \frac{1}{r_2},$$

*we have $(f * g)_\rho \in \mathcal{N}(p, q, s)$ if $f \in A_{\rho-1}^{r_1}$ and $g \in A_{\rho-1}^{r_2}$.*

Proof. By Proposition 4.2 and Lemma 4.8, we have

$$\|(f * g)_\rho\| \lesssim \|(f * g)_\rho\|_{\frac{p(n+\rho)}{q}, (\rho-1)} \leq \|f\|_{r_1, (\rho-1)} \|g\|_{r_2, (\rho-1)},$$

which implies the desired result. \square

Corollary 4.11. *Let $p \geq 1, q > 0$ and $s > \max\{0, 1 - \frac{q}{n}\}$. Then $(f * g)_{q+ns-n} \in \mathcal{N}(p, q, s)$ if $f \in \mathcal{N}(p, q, s)$ and $g \in A_{q+ns-n-1}^{\frac{p(q+ns)}{p(q+ns)-ns}}$.*

Proof. By Proposition 4.10, we have

$$\begin{aligned} \|(f * g)_{(q+ns-n)}\| &\lesssim \|f\|_{p, (q+ns-n-1)} \|g\|_{\frac{p(q+ns)}{p(q+ns)-ns}, q+ns-n-1} \\ &\lesssim \|f\| \|g\|_{\frac{p(q+ns)}{p(q+ns)-ns}, q+ns-n-1}. \end{aligned}$$

Hence, we get the desired result. \square

The following lemma gives an estimation of the term $M_\alpha(r, (f * g)_d)$, which gives us another description of Hadamard products via the embedding relation between $\mathcal{N}(p, q, s)$ and H_β^α .

Lemma 4.12. *Let $\alpha \geq 1, r \in [0, 1), d > 0$ and $f, g \in H(\mathbb{B})$, then*

$$M_\alpha(r, (f * g)_d) \leq M_\alpha(\sqrt{r}, g) \|f\|_{1, d-1}.$$

Proof. Without the loss of generality, we assume that $\|f\|_{1, d-1} < \infty$. Thus, by (4.14), [1, Proposition 3.1, (i)] and the fact that the integral mean of subharmonic function over sphere is an increasing function of multi-radius, we have

$$\begin{aligned} M_\alpha(r, (f * g)_d) &= \left(\int_{\mathbb{S}} |(f * g)_d(r\xi)|^\alpha d\sigma(\xi) \right)^{\frac{1}{\alpha}} \\ &= \left(\int_{\mathbb{S}} |(f * g)_d(\sqrt{r} \cdot \sqrt{r}\xi)|^\alpha d\sigma(\xi) \right)^{\frac{1}{\alpha}} \\ &= \left(\int_{\mathbb{S}} \left| \int_{\mathbb{B}} f(\sqrt{r}w) g(\sqrt{r}\xi \cdot \bar{w}) dV_{d-1}(w) \right|^\alpha d\sigma(\xi) \right)^{\frac{1}{\alpha}} \\ &\leq \int_{\mathbb{B}} |f(\sqrt{r}w)| \left(\int_{\mathbb{S}} |g(\sqrt{r}\xi \cdot \bar{w})|^\alpha d\sigma(\xi) \right)^{\frac{1}{\alpha}} dV_{d-1}(w) \\ &\quad \text{(By Minkowski's inequality)} \\ &\leq \int_{\mathbb{B}} |f(\sqrt{r}w)| \left(\int_{\mathbb{S}} |g(\sqrt{r}\xi)|^\alpha d\sigma(\xi) \right)^{\frac{1}{\alpha}} dV_{d-1}(w) \\ &\quad \text{(Since } |g(z)|^\alpha \text{ is subharmonic)} \\ &\leq M_\alpha(\sqrt{r}, g) \|f\|_{1, d-1}. \end{aligned}$$

\square

Proposition 4.13. *Let $d > 0$ and p, q, s, α satisfy the conditions in Proposition 4.5. Then we have $(f * g)_d \in \mathcal{N}(p, q, s)$ if $f \in A_{d-1}^1$ and $g \in H_{\frac{q}{p} - \frac{n}{\alpha}}^\alpha$.*

Proof. For any $f \in A_{d-1}^1$ and $g \in H_{\frac{q}{p}-\frac{n}{\alpha}}^\alpha$, by Proposition 4.5 and Lemma 4.12, we have

$$\begin{aligned}
\|(f * g)_d\| &\lesssim \sup_{0 < r < 1} (1-r)^{\frac{q}{p}-\frac{n}{\alpha}} M_\alpha(r, (f * g)_d) \\
&\leq \|f\|_{1,d-1} \cdot \sup_{0 < r < 1} (1-r)^{\frac{q}{p}-\frac{n}{\alpha}} M_\alpha(\sqrt{r}, g) \\
&\lesssim \|f\|_{1,d-1} \cdot \sup_{0 < r < 1} (1-\sqrt{r})^{\frac{q}{p}-\frac{n}{\alpha}} M_\alpha(\sqrt{r}, g) \\
&= \|f\|_{1,d-1} \|g\|_{H_{\frac{q}{p}-\frac{n}{\alpha}}^\alpha}.
\end{aligned}$$

which implies $(f * g)_d \in \mathcal{N}(p, q, s)$. \square

5. CHARACTERIZATIONS OF $\mathcal{N}(p, q, s)$ -TYPE SPACES

Recall that by Proposition 4.1, f belongs to $\mathcal{N}(p, q, s)$ is equivalent to $d\mu = d\mu_{f,p,q,s}(z) = |f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z)$ is an (ns) -Carleson measure. In this section, we extend this result and establish several characterizations of the $\mathcal{N}(p, q, s)$ -norm.

Before we formulate our main result, let us recall several notations. For $f \in H(\mathbb{B})$, $z \in \mathbb{B}$. Let

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z) \right)$$

denote the complex gradient of f and let $\tilde{\nabla} f$ denote the invariant gradient of \mathbb{B} , i.e., $(\tilde{\nabla} f)(z) = \nabla(f \circ \Phi_z)(0)$. Moreover, we write

$$Rf(z) = \sum_{k=1}^n z_k \frac{\partial f}{\partial z_k}(z)$$

as the radial derivative of f (see, e.g., [25]) and for $1 \leq i, j \leq n$,

$$T_{i,j}f(z) = \bar{z}_j \frac{\partial f}{\partial z_i} - \bar{z}_i \frac{\partial f}{\partial z_j}$$

as the tangential derivative of f (see, e.g., [7]). We need the following lemmas.

Lemma 5.1. *Let $\xi \in \mathbb{S}$ and $0 < \delta < 1$, then there exists some $M > 0$, which is independent of δ , such that*

$$\bigcup_{z \in Q_\delta(\xi)} D\left(z, \frac{1}{4}\right) \subseteq Q_{M\delta}(\xi)$$

Proof. Take $w \in \bigcup_{z \in Q_\delta(\xi)} D(z, \frac{1}{4})$, which implies $w \in D(z, \frac{1}{4})$ for some $z \in Q_\delta(\xi)$. Since $|\Phi_w(z)| < \frac{1}{4}$, by [25, Proposition 1.21 and Lemma 2.20], there exists some $M' > 0$ independent of z and w , such that

$$(5.1) \quad \frac{1}{M'} \leq \frac{1 - |w|^2}{1 - |z|^2} \leq M'.$$

Hence, we have

$$\begin{aligned} |1 - \langle w, \xi \rangle|^{1/2} &\leq |1 - \langle w, z \rangle|^{1/2} + |1 - \langle z, \xi \rangle|^{1/2} \\ &\leq \delta^{1/2} + \left(\frac{(1 - |z|^2)(1 - |w|^2)}{1 - |\Phi_w(z)|^2} \right)^{\frac{1}{4}} \\ &\leq \delta^{1/2} + 2M'^{1/4} \delta^{1/2}. \end{aligned}$$

The result follows by taking $M = (1 + 2M'^{1/4})^2$. \square

Lemma 5.2. *For $f \in H(\mathbb{B})$, there exists some constant $C > 0$, such that*

$$|Rf(z)| \leq \frac{C}{(1 - |z|^2)^{1/2}} \int_{D(z, \frac{1}{4})} \sum_{i < j} |T_{i,j}f(w)| d\lambda(w), \quad \forall z \in \mathbb{B}$$

Proof. The proof of this lemma is a simple modification of [6, Lemma 2] and hence we omit it here. \square

We have the following result.

Theorem 5.3. *Let $f \in H(\mathbb{B})$ and $p \geq 1, q > 0$ and $s > \max\{0, 1 - \frac{q}{n}\}$. The following statements are equivalent:*

- (1) $f \in \mathcal{N}(p, q, s)$ or equivalently, $d\mu_1 = |f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z)$ is an (ns) -Carleson measure;
- (2) $d\mu_2 = |\nabla f(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z)$ is an (ns) -Carleson measure;
- (3) $d\mu_3 = |\tilde{\nabla} f(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z)$ is an (ns) -Carleson measure;
- (4) $d\mu_4 = |Rf(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z)$ is an (ns) -Carleson measure;
- (5) $d\mu_5 = \left(\sum_{i < j} |T_{i,j}f(z)|^p \right) (1 - |z|^2)^{\frac{p}{2}+q+ns} d\lambda(z)$ is an (ns) -Carleson measure.

Proof. Note that the equivalence between (1) and (2) follows from [28, Theorem 3.2] and [27, Theorem 45]. Moreover, since $(1 - |z|^2)|Rf(z)| \leq (1 - |z|^2)|\nabla f(z)| \leq |\tilde{\nabla} f(z)|, z \in \mathbb{B}$, it is clear that (3) \implies (2) \implies (4).

Furthermore, the identity

$$|z|^2 |\tilde{\nabla} f(z)|^2 = (1 - |z|^2) \left((1 - |z|^2) |Rf(z)|^2 + \sum_{i < j} |T_{i,j} f(z)|^2 \right)$$

implies that (3) follows from (4) and (5). Therefore, it suffices to show the equivalence between (4) and (5).

(5) \implies (4). Suppose $d\mu_5$ is an (ns) -Carleson measure. First we note that for any $z \in \mathbb{B}$, the integral

$$(5.2) \quad \int_{D(z, \frac{1}{4})} d\lambda(w) < K,$$

for some K independent of the choice of z . Indeed, by [18, 2.2.7], we have

$$\begin{aligned} \int_{D(z, \frac{1}{4})} d\lambda(w) &= \int_{D(z, \frac{1}{4})} \frac{1}{(1 - |w|^2)^{n+1}} dV(w) \\ &\simeq \frac{1}{(1 - |z|^2)^{n+1}} \int_{D(z, \frac{1}{4})} dV(w) \quad (\text{by (5.1)}) \\ &\simeq \frac{1}{(1 - |z|^2)^{n+1}} \cdot (1 - |z|^2)^{n+1} = 1. \end{aligned}$$

By Lemma 5.1, Lemma 5.2, (5.1) and (5.2), for any $\xi \in \mathbb{S}$ and $\delta \in (0, 1)$,

$$\begin{aligned} &\int_{Q_\delta(\xi)} |Rf(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z) \\ &\lesssim \int_{Q_\delta(\xi)} \left(\int_{D(z, \frac{1}{4})} \sum_{i < j} |T_{i,j} f(w)| d\lambda(w) \right)^p (1 - |z|^2)^{\frac{p}{2}+q+ns} d\lambda(z) \\ &\lesssim \int_{Q_\delta(\xi)} \sum_{i < j} \left(\int_{D(z, \frac{1}{4})} |T_{i,j} f(w)| d\lambda(w) \right)^p (1 - |z|^2)^{\frac{p}{2}+q+ns} d\lambda(z) \\ &\lesssim \int_{Q_\delta(\xi)} \left(\sum_{i < j} \int_{D(z, \frac{1}{4})} |T_{i,j} f(w)|^p d\lambda(w) \right) (1 - |z|^2)^{\frac{p}{2}+q+ns} d\lambda(z) \\ &\leq \int_{\mathbb{B}} \sum_{i < j} |T_{i,j} f(w)|^p \chi_{\bigcup_{z \in Q_\delta(\xi)} D(z, \frac{1}{4})}(w) \left(\int_{Q_\delta(\xi)} \chi_{D(z, \frac{1}{4})}(z) (1 - |z|^2)^{\frac{p}{2}+q+ns} d\lambda(z) \right) d\lambda(w) \end{aligned}$$

$$\begin{aligned}
&\simeq \int_{\mathbb{B}} (1 - |w|^2)^{\frac{p}{2}+q+ns} \sum_{i < j} |T_{i,j}f(w)|^p \chi_{\bigcup_{z \in Q_\delta(\xi)} D(z, \frac{1}{4})}(w) \left(\int_{Q_\delta(\xi)} \chi_{D(z, \frac{1}{4})}(z) d\lambda(z) \right) d\lambda(w) \\
&\lesssim \int_{\mathbb{B}} (1 - |w|^2)^{\frac{p}{2}+q+ns} \sum_{i < j} |T_{i,j}f(w)|^p \chi_{\bigcup_{z \in Q_\delta(\xi)} D(z, \frac{1}{4})}(w) d\lambda(w) \\
&\leq \int_{Q_{M\delta}(\xi)} (1 - |w|^2)^{\frac{p}{2}+q+ns} \sum_{i < j} |T_{i,j}f(w)|^p d\lambda(w) \\
&\lesssim \delta^{ns} \quad (\text{since } d\mu_5 \text{ is an } (ns)\text{-Carleson measure}).
\end{aligned}$$

Hence, we get the desired result.

(4) \implies (5). Suppose $d\mu_4$ is an (ns) -Carleson measure. From

$$f(z) - f(0) = \int_0^1 \frac{d}{dt} f(tz) dt = \int_0^1 \frac{Rf(tz)}{t} dt,$$

we see that for $1 \leq i, j \leq n$,

$$T_{i,j}f(z) = \int_0^1 \frac{T_{i,j}(Rf(tz))}{t} dt = \int_0^1 \frac{(T_{i,j}Rf)(tz)}{t} dt.$$

Hence, it suffices to prove for each $1 \leq i, j \leq n$,

$$\left| \int_0^1 \frac{(T_{i,j}Rf)(tz)}{t} dt \right|^p (1 - |z|^2)^{\frac{p}{2}+q+ns} d\lambda(z)$$

is an (ns) -Carleson measure.

Note that by [25, Corollary 5.24] and the fact that $\frac{p}{2}+q+ns-n-1 > -1$, we have, for any $\xi \in \mathbb{S}$ and $0 < \delta < 1$,

$$\begin{aligned}
&\int_{Q_r(\xi)} \left| \int_0^{1/2} \frac{(T_{i,j}Rf)(tz)}{t} dt \right|^p (1 - |z|^2)^{\frac{p}{2}+q+ns} d\lambda(z) \\
&\lesssim \int_{Q_r(\xi)} (1 - |z|^2)^{\frac{p}{2}+q+ns-n-1} dV(z) \\
&\simeq r^{\frac{p}{2}+q+ns} \leq r^{ns}
\end{aligned}$$

Thus, we only need to show for $1 \leq i, j \leq n$,

$$\left(\int_{1/2}^1 |(T_{i,j}Rf)(tz)| dt \right)^p (1 - |z|^2)^{\frac{p}{2}+q+ns} d\lambda(z)$$

is an (ns) -Carleson measure.

By the proof of [6, Theorem 1], for any $\gamma \geq 0$, we have

$$(5.3) \quad \int_{1/2}^1 |(T_{i,j}Rf)(tw)| dt \lesssim \int_{\mathbb{B}} \frac{(1 - |z|^2)^\gamma |Rf(z)|}{|1 - \langle z, w \rangle|^{n+\gamma+\frac{1}{2}}} dV(z), \quad w \in \mathbb{B}.$$

Now we consider two cases.

Case I: $p > 1$. Let p' be the conjugate of p , take and fix two positive numbers γ and ρ , such that

$$\max \left\{ 0, 1 - \frac{p'}{2} \right\} < 2p'\rho < 1, \gamma > \max \{ (p + p')\rho, p + q + ns - n - 1 - p\rho \}.$$

Then for $w \in \mathbb{B}$, by [25, Theorem 1.12], we have

$$\begin{aligned} & \left(\int_{\mathbb{B}} \frac{(1 - |z|^2)^\gamma |Rf(z)|}{|1 - \langle z, w \rangle|^{n+\gamma+\frac{1}{2}}} dV(z) \right)^p = \left(\int_{\mathbb{B}} \frac{(1 - |z|^2)^{\frac{\gamma}{p}} (1 - |z|^2)^{\frac{\gamma}{p'}} |Rf(z)|}{|1 - \langle z, w \rangle|^{n+\gamma+\frac{1}{2}}} dV(z) \right)^p \\ &= \left(\int_{\mathbb{B}} \frac{(1 - |z|^2)^{\frac{\gamma}{p} + \rho} (1 - |z|^2)^{\frac{\gamma}{p'} - \rho} |Rf(z)|}{|1 - \langle z, w \rangle|^{\frac{n+\gamma}{p} - \rho} |1 - \langle z, w \rangle|^{\frac{n+\gamma}{p'} + \rho + \frac{1}{2}}} dV(z) \right)^p \\ &\leq \left(\int_{\mathbb{B}} \frac{(1 - |z|^2)^{\gamma+p\rho} |Rf(z)|^p}{|1 - \langle z, w \rangle|^{n+\gamma-p\rho}} dV(z) \right) \left(\int_{\mathbb{B}} \frac{(1 - |z|^2)^{\gamma-p'\rho}}{|1 - \langle z, w \rangle|^{n+\gamma+p'\rho+\frac{p'}{2}}} dV(z) \right)^{p/p'} \\ &\simeq \left(\int_{\mathbb{B}} \frac{(1 - |z|^2)^{\gamma+p\rho} |Rf(z)|^p}{|1 - \langle z, w \rangle|^{n+\gamma-p\rho}} dV(z) \right) \cdot \left((1 - |w|^2)^{1-2p'\rho-\frac{p'}{2}} \right)^{p-1} \\ &= \left(\int_{\mathbb{B}} \frac{(1 - |z|^2)^{\gamma+p\rho} |Rf(z)|^p}{|1 - \langle z, w \rangle|^{n+\gamma-p\rho}} dV(z) \right) \cdot (1 - |w|^2)^{\frac{p}{2}-2p\rho-1} \end{aligned}$$

Thus, for any $\xi \in \mathbb{S}$ and $rl \in (0, 1)$,

$$\begin{aligned} & \int_{Q_r(\xi)} \left(\int_{1/2}^1 |(T_{i,j} Rf)(tw)| dt \right)^p (1 - |w|^2)^{\frac{p}{2}+q+ns} d\lambda(w) \\ &\lesssim \int_{Q_r(\xi)} \left(\int_{\mathbb{B}} \frac{(1 - |z|^2)^\gamma |Rf(z)|}{|1 - \langle z, w \rangle|^{n+\gamma+\frac{1}{2}}} dV(z) \right)^p (1 - |w|^2)^{\frac{p}{2}+q+ns} d\lambda(w) \\ &\lesssim \int_{Q_r(\xi)} \left(\int_{\mathbb{B}} \frac{(1 - |z|^2)^{\gamma+p\rho} |Rf(z)|^p}{|1 - \langle z, w \rangle|^{n+\gamma-p\rho}} dV(z) \right) \cdot (1 - |w|^2)^{p+q+ns-2p\rho-1} d\lambda(w) \\ &= \int_{Q_r(\xi)} \left(\int_{Q_{2r}(\xi)} \frac{(1 - |z|^2)^{\gamma+p\rho} |Rf(z)|^p}{|1 - \langle z, w \rangle|^{n+\gamma-p\rho}} dV(z) \right) (1 - |w|^2)^{p+q+ns-2p\rho-1} d\lambda(w) \\ &\quad + \sum_{j=1}^{\infty} \int_{Q_r(\xi)} \left\{ \left(\int_{Q_{2^{j+1}r}(\xi) \setminus Q_{2^j r}(\xi)} \frac{(1 - |z|^2)^{\gamma+p\rho} |Rf(z)|^p}{|1 - \langle z, w \rangle|^{n+\gamma-p\rho}} dV(z) \right) \right. \\ &\quad \left. (1 - |w|^2)^{p+q+ns-2p\rho-1} d\lambda(w) \right\} \\ &= I_1 + I_2. \end{aligned}$$

- **Estimation of I_1 .**

Since $2p'\rho < 1$ and $s > 1 - \frac{q}{n}$, it follows that

$$p + q + ns - n - 2p\rho - 2 > p - 2p\rho - 2 > -1.$$

Hence, by [25, Theorem 1.12] and Fubini's theorem, we have

$$\begin{aligned} I_1 &= \int_{Q_{2r}(\xi)} (1 - |z|^2)^{\gamma+p\rho} |Rf(z)|^p \left(\int_{Q_r(\xi)} \frac{(1 - |w|^2)^{p+q+ns-n-2p\rho-2}}{|1 - \langle z, w \rangle|^{n+t-p\rho}} dV(w) \right) dV(z) \\ &\leq \int_{Q_{2r}(\xi)} (1 - |z|^2)^{\gamma+p\rho} |Rf(z)|^p \left(\int_{\mathbb{B}} \frac{(1 - |w|^2)^{p+q+ns-n-2p\rho-2}}{|1 - \langle z, w \rangle|^{n+t-p\rho}} dV(w) \right) dV(z) \\ &\simeq \int_{Q_{2r}(\xi)} |Rf(z)|^p (1 - |z|^2)^{\gamma+p\rho+p+q+ns-\gamma-p\rho-n-1} dV(z) \\ &= \int_{Q_{2r}(\xi)} |Rf(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z) \\ &\lesssim r^{ns}. \end{aligned}$$

• **Estimation of I_2 .**

First we note that for $w \in Q_r(\xi)$ and $z \in Q_{2^{j+1}r}(\xi) \setminus Q_{2^j r}(\xi)$, $j \in \mathbb{N}$, we have

$$\begin{aligned} |1 - \langle z, w \rangle|^{1/2} &\geq |1 - \langle z, \xi \rangle|^{1/2} - |1 - \langle w, \xi \rangle|^{1/2} \\ &\geq (2^{j/2} - 1)r^{1/2}, \end{aligned}$$

which implies for $w \in Q_r(\xi)$ and $z \in Q_{2^{j+1}r}(\xi) \setminus Q_{2^j r}(\xi)$, $j \in \mathbb{N}$, we have $|1 - \langle z, w \rangle| \geq \frac{2^j r}{100}$. Since $\gamma + p\rho > p + q + ns - n - 1$, we put $\beta = \gamma + p\rho - p - q - ns + n + 1 > 0$, and hence $n + \gamma - p\rho = p + q + ns + \beta - 2p\rho - 1$. Moreover, since $2p'\rho < 1$, we have $2p\rho + 1 - p - q < 0$.

Thus, using the fact that $p + q + ns - 2p\rho - n - 2 > -1$ and [25, Corollary 5.24], it follows that

$$\begin{aligned} I_2 &= \sum_{j=1}^{\infty} \left\{ \int_{Q_r(\xi)} \left(\int_{Q_{2^{j+1}r}(\xi) \setminus Q_{2^j r}(\xi)} \frac{(1 - |z|^2)^{\beta}}{|1 - \langle z, w \rangle|^{\beta}} \cdot \frac{(1 - |z|^2)^{p+q+ns-n-1} |Rf(z)|^p}{|1 - \langle z, w \rangle|^{p+q+ns-2p\rho-1}} dV(z) \right) \right. \\ &\quad \left. (1 - |w|^2)^{p+q+ns-2p\rho-1} d\lambda(w) \right\} \\ &\lesssim \sum_{j=1}^{\infty} \left\{ \int_{Q_r(\xi)} \left(\int_{Q_{2^{j+1}r}(\xi) \setminus Q_{2^j r}(\xi)} \frac{(1 - |z|^2)^{p+q+ns-n-1} |Rf(z)|^p}{|1 - \langle z, w \rangle|^{p+q+ns-2p\rho-1}} dV(z) \right) \right. \\ &\quad \left. (1 - |w|^2)^{p+q+ns-2p\rho-1} d\lambda(w) \right\} \end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{j=1}^{\infty} \left\{ (2^j r)^{2p\rho+1-p-q-ns} \int_{Q_r(\xi)} (1-|w|^2)^{p+q+ns-2p\rho-1} \right. \\
&\quad \left. \left(\int_{Q_{2^{j+1}r}(\xi)} (1-|z|^2)^{p+q+ns} |Rf(z)|^p d\lambda(z) \right) d\lambda(w) \right\} \\
&\lesssim \sum_{j=1}^{\infty} \left\{ (2^j r)^{2p\rho+1-p-q-ns} (2^{j+1}r)^{ns} \int_{Q_r(\xi)} (1-|w|^2)^{p+q+ns-2p\rho-1} d\lambda(z) \right\} \\
&\simeq \sum_{j=1}^{\infty} (2^j r)^{2p\rho+1-p-q-ns} (2^{j+1}r)^{ns} r^{p+q+ns-2p\rho-1} \\
&\simeq \left(\sum_{j=1}^{\infty} 2^{j(2p\rho+1-p-q)} \right) r^{ns} \lesssim r^{ns}.
\end{aligned}$$

Finally, combining the estimations of I_1 and I_2 , we get the desired result.

Case II: $p = 1$.

Take and fix a positive number γ , such that $\gamma > q + ns - n$, and put $\beta = \gamma - q - ns + n$, which implies that

$$n + \gamma + \frac{1}{2} = \beta + q + ns + \frac{1}{2}.$$

For any $\xi \in \mathbb{S}$ and $r \in (0, 1)$, we have

$$\begin{aligned}
&\int_{Q_r(\xi)} \left(\int_{1/2}^1 |(T_{i,j}Rf)(tw)| dt \right) (1-|w|^2)^{\frac{1}{2}+q+ns} d\lambda(w) \\
&\lesssim \int_{Q_r(\xi)} \left(\int_{\mathbb{B}} \frac{(1-|z|^2)^\gamma |Rf(z)|}{|1-\langle z, w \rangle|^{n+\gamma+\frac{1}{2}}} dV(z) \right) (1-|w|^2)^{\frac{1}{2}+q+ns} d\lambda(w) \\
&= \int_{Q_r(\xi)} (1-|w|^2)^{\frac{1}{2}+q+ns} \left(\int_{Q_{2r}(\xi)} \frac{(1-|z|^2)^\gamma |Rf(z)|}{|1-\langle z, w \rangle|^{n+\gamma+\frac{1}{2}}} dV(z) \right) d\lambda(w) \\
&\quad + \sum_{j=1}^{\infty} \int_{Q_r(\xi)} \left\{ \left(\int_{Q_{2^{j+1}r}(\xi) \setminus Q_{2^j r}(\xi)} \frac{(1-|z|^2)^\gamma |Rf(z)|}{|1-\langle z, w \rangle|^{n+\gamma+\frac{1}{2}}} dV(z) \right) \right. \\
&\quad \left. (1-|w|^2)^{\frac{1}{2}+q+ns} d\lambda(w) \right\} \\
&= J_1 + J_2.
\end{aligned}$$

• **Estimation of J_1 .**

By Fubini's theorem and [25, Theorem 1.12], we have

$$\begin{aligned}
 J_1 &= \int_{Q_r(\xi)} (1 - |w|^2)^{q+ns-n-\frac{1}{2}} \left(\int_{Q_{2r}(\xi)} \frac{(1 - |z|^2)^\gamma |Rf(z)|}{|1 - \langle z, w \rangle|^{n+\gamma+\frac{1}{2}}} dV(z) \right) dV(w) \\
 &\leq \int_{Q_{2r}(\xi)} |Rf(z)| (1 - |z|^2)^\gamma \left(\int_{\mathbb{B}} \frac{(1 - |w|^2)^{q+ns-n-\frac{1}{2}}}{|1 - \langle z, w \rangle|^{n+\gamma+\frac{1}{2}}} dV(w) \right) dV(z) \\
 &\simeq \int_{Q_{2r}(\xi)} |Rf(z)| (1 - |z|^2)^\gamma (1 - |z|^2)^{q+ns-n-\gamma} dV(z) \\
 &= \int_{Q_{2r}(\xi)} |Rf(z)| (1 - |z|^2)^{1+q+ns} d\lambda(z) \lesssim r^{ns}.
 \end{aligned}$$

• **Estimation of J_2 .**

By our choice of γ and [25, Corollary 5.24], we have

$$\begin{aligned}
 J_2 &= \sum_{j=1}^{\infty} \int_{Q_r(\xi)} \left\{ \left(\int_{Q_{2^{j+1}r}(\xi) \setminus Q_{2^j r}(\xi)} \frac{(1 - |z|^2)^\beta}{|1 - \langle z, w \rangle|^\beta} \cdot \frac{(1 - |z|^2)^{q+ns-n} |Rf(z)|}{|1 - \langle z, w \rangle|^{q+ns+\frac{1}{2}}} dV(z) \right) \right. \\
 &\quad \left. (1 - |w|^2)^{\frac{1}{2}+q+ns} d\lambda(w) \right\} \\
 &\lesssim \sum_{j=1}^{\infty} \int_{Q_r(\xi)} \left\{ \left(\int_{Q_{2^{j+1}r}(\xi) \setminus Q_{2^j r}(\xi)} \frac{(1 - |z|^2)^{q+ns-n} |Rf(z)|}{|1 - \langle z, w \rangle|^{q+ns+\frac{1}{2}}} dV(z) \right) \right. \\
 &\quad \left. (1 - |w|^2)^{\frac{1}{2}+q+ns} d\lambda(w) \right\} \\
 &\lesssim \sum_{j=1}^{\infty} \left\{ (2^j r)^{-q-ns-\frac{1}{2}} \int_{Q_r(\xi)} (1 - |w|^2)^{\frac{1}{2}+q+ns} \right. \\
 &\quad \left. \left(\int_{Q_{2^{j+1}r}(\xi)} |Rf(z)| (1 - |z|^2)^{1+q+ns} d\lambda(z) \right) d\lambda(w) \right\} \\
 &\lesssim \sum_{j=1}^{\infty} (2^j r)^{-q-ns-\frac{1}{2}} (2^{j+1} r)^{ns} \int_{Q_r(\xi)} (1 - |w|^2)^{\frac{1}{2}+q+ns} d\lambda(w) \\
 &\simeq \sum_{j=1}^{\infty} (2^j r)^{-q-ns-\frac{1}{2}} (2^{j+1} r)^{ns} r^{\frac{1}{2}+q+ns} \lesssim r^{ns}.
 \end{aligned}$$

The desired result follows from the estimations on J_1 and J_2 . \square

Correspondingly, we have the following result for $\mathcal{N}^0(p, q, s)$ -spaces and vanishing Carleson measure.

Theorem 5.4. *Let $f \in H(\mathbb{B})$, $p \geq 1$, $q > 0$ and $s > \max\{0, 1 - \frac{q}{n}\}$. The following statements are equivalent:*

- (1) $f \in \mathcal{N}^0(p, q, s)$ or equivalently, $d\mu_1 = |f(z)|^p(1 - |z|^2)^{q+ns}d\lambda(z)$ is a vanishing (ns) -Carleson measure;
- (2) $d\mu_2 = |\nabla f(z)|^p(1 - |z|^2)^{p+q+ns}d\lambda(z)$ is a vanishing (ns) -Carleson measure;
- (3) $d\mu_3 = |\tilde{\nabla} f(z)|^p(1 - |z|^2)^{q+ns}d\lambda(z)$ is a vanishing (ns) -Carleson measure;
- (4) $d\mu_4 = |Rf(z)|^p(1 - |z|^2)^{p+q+ns}d\lambda(z)$ is a vanishing (ns) -Carleson measure;
- (5) $d\mu_5 = \left(\sum_{i < j} |T_{i,j}f(z)|^p \right) (1 - |z|^2)^{\frac{p}{2}+q+ns}d\lambda(z)$ is a vanishing (ns) -Carleson measure.

Proof. The proof of this theorem is a simple modification of Theorem 5.3 and hence we omit it here. \square

Moreover, combining Theorem 5.3, [27, Theorem 45] and [28, Theorem 3.2], we have the following characterizations of $\mathcal{N}(p, q, s)$ -norm. In particular, we have:

Corollary 5.5. *Let $f \in H(\mathbb{B})$, $p \geq 1$, $q > 0$ and $s > \max\{0, 1 - \frac{q}{n}\}$. The following quantities are equivalent.*

(1)

$$I_1 = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z);$$

(2)

$$I_2 = |f(0)|^p + \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\nabla f(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z);$$

(3)

$$I_3 = |f(0)|^p + \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |Rf(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z);$$

(4)

$$I_4 = |f(0)|^p + \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z).$$

The following description via higher radial derivative is straightforward from the above corollary.

Corollary 5.6. *Let $f \in H(\mathbb{B})$, $p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{n}\}$ and $m \in \mathbb{N}$. Then $f \in \mathcal{N}(p, q, s)$ if and only if*

$$\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |R^m f(z)|^p (1 - |z|^2)^{mp+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty.$$

As an application of Corollary 5.5, we study some other new derivative-free, mixture and oscillation characterizations to $\mathcal{N}(p, q, s)$ -spaces, whose idea comes from [9].

We need the following lemma, which was proved in [16].

Lemma 5.7. *Suppose $\alpha > -1, p > 0, 0 \leq \beta < p + 2$ and $f \in H(\mathbb{B})$. Then $f \in A_{\alpha}^p$ if and only if*

$$K(f) = \int_{\mathbb{B}} |f(z)|^{p-\beta} |\tilde{\nabla} f(z)|^{\beta} dV_{\alpha}(z) < \infty.$$

Moreover, the quantities $\|f\|_{p,\alpha}^p$ and $|f(0)|^p + K(f)$ are comparable for $f \in H(\mathbb{B})$.

Theorem 5.8. *Suppose $f \in H(\mathbb{B}), p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{n}\}, 0 \leq \beta < p + 2$ and $\alpha > q + ns - n - 1$. Then $f \in \mathcal{N}(p, q, s)$ if and only if*

$$M = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f(z) - f(w)|^{p-\beta}}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} |\tilde{\nabla} f(z)|^{\beta} (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} dV_{\alpha}(w) dV_{\alpha}(z) < \infty.$$

Proof. • **Necessity.** Suppose $M < \infty$. Note that by $|\tilde{\nabla}(f \circ \Phi_w)(z)| = |\tilde{\nabla} f(\Phi_w(z))|$ and $\alpha > q + ns - n - 1 > -1$, we have

$$M = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} \left(\int_{\mathbb{B}} |F_w(z)|^{p-\beta} |\tilde{\nabla} F_w(z)|^{\beta} dV_{\alpha}(z) \right) d\lambda(w),$$

where $F_w = f \circ \Phi_w - f(w), w \in \mathbb{B}$. By Lemma 5.7,

$$M \simeq \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} \left(\int_{\mathbb{B}} |F_w(z)|^p dV_{\alpha}(z) \right) d\lambda(w).$$

Note that by [25, Lemma 2.4], we have, for any $w \in \mathbb{B}$,

$$\begin{aligned} |\tilde{\nabla} f(w)|^p &= |\nabla(f \circ \Phi_w)(0)|^p = |\nabla(f \circ \Phi_w - f(w))(0)|^p \\ &\lesssim \int_{\mathbb{B}} |f \circ \Phi_w(z) - f(w)|^p dV_{\alpha}(z) = \int_{\mathbb{B}} |F_w(z)|^p dV_{\alpha}(z). \end{aligned}$$

Hence

$$\infty > M \gtrsim \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\tilde{\nabla} f(w)|^p (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} d\lambda(w),$$

which, by Corollary 5.5, implies $f \in \mathcal{N}(p, q, s)$.

• **Sufficiency.** Suppose $f \in \mathcal{N}(p, q, s)$. First by Lemma 5.7, we see that the quantities

$$\int_{\mathbb{B}} |F_w(z)|^{p-\beta} |\tilde{\nabla} F_w(z)|^\beta dV_\alpha(z) \text{ and } \int_{\mathbb{B}} |\tilde{\nabla} F_w(z)|^p dV_\alpha(z)$$

are comparable. Hence, by (5.4), we have

$$\begin{aligned} M &= \int_{\mathbb{B}} (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} \left(\int_{\mathbb{B}} |F_w(z)|^{p-\beta} |\tilde{\nabla} F_w(z)|^\beta dV_\alpha(z) \right) d\lambda(w) \\ &\lesssim \int_{\mathbb{B}} \left(\int_{\mathbb{B}} |\tilde{\nabla} F_w(z)|^p dV_\alpha(z) \right) (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} d\lambda(w) \\ &= \int_{\mathbb{B}} \left(\int_{\mathbb{B}} |\tilde{\nabla} f(\Phi_w(u))|^p dV_\alpha(u) \right) (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} d\lambda(w) \\ &= \int_{\mathbb{B}} \left(\int_{\mathbb{B}} |\tilde{\nabla} f(z)|^p (1 - |\Phi_w(z)|^2)^{n+1+\alpha} d\lambda(z) \right) \\ &\quad (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} d\lambda(w) \\ &\leq \int_{\mathbb{B}} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^p (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \cdot I, \end{aligned}$$

where

$$\begin{aligned} I &= \sup_{a, z \in \mathbb{B}} \int_{\mathbb{B}} \frac{(1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns}}{(1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns}} (1 - |\Phi_w(z)|^2)^{n+1+\alpha} d\lambda(w) \\ &= \sup_{a, z \in \mathbb{B}} \int_{\mathbb{B}} \frac{(1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns}}{(1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns}} (1 - |\Phi_z(w)|^2)^{n+1+\alpha} d\lambda(w) \\ &= \sup_{a, z \in \mathbb{B}} \int_{\mathbb{B}} \frac{(1 - |\Phi_z(u)|^2)^q (1 - |\Phi_a(\Phi_z(u))|^2)^{ns}}{(1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns}} (1 - |u|^2)^{n+1+\alpha} d\lambda(u) \\ &\simeq \sup_{a, z \in \mathbb{B}} \int_{\mathbb{B}} \frac{(1 - |u|^2)^q (1 - |u|^{2n})^{ns}}{|1 - \langle u, a \rangle|^{2q} |1 - \langle \Phi_a(z), u \rangle|^{2ns}} dV_\alpha(u) \\ &\leq \sup_{a, z \in \mathbb{B}} \left\{ \left(\int_{\mathbb{B}} \frac{(1 - |u|^2)^{q+ns}}{|1 - \langle u, a \rangle|^{2(q+ns)}} dV_\alpha(u) \right)^{\frac{q}{q+ns}} \right. \\ &\quad \left. \cdot \left(\int_{\mathbb{B}} \frac{(1 - |u|^2)^{q+ns}}{|1 - \langle u, \Phi_z(a) \rangle|^{2(q+ns)}} dV_\alpha(u) \right)^{\frac{ns}{q+ns}} \right\} \\ &< \infty. \end{aligned}$$

Here, we use the fact that $q + ns - n - 1 - \alpha < 0$ in the last inequality.

Thus, we get

$$\int_{\mathbb{B}} (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} \left(\int_{\mathbb{B}} |F_w(z)|^{p-\beta} |\tilde{\nabla} F_w(z)|^\beta dV_\alpha(z) \right) d\lambda(w) \lesssim \|f\|^p,$$

which implies the desired result. \square

In particular, taking $\beta = 0$, we get the following result.

Theorem 5.9. *Suppose $f \in H(\mathbb{B})$, $p \geq 1$, $q > 0$, $s > \max\{0, 1 - \frac{q}{n}\}$ and $\alpha > q + ns - n - 1$. Then $f \in \mathcal{N}(p, q, s)$ if and only if*

$$\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f(z) - f(w)|^p}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} (1 - |w|^2)^q (1 - |\Phi_a(w)|^2)^{ns} dV_\alpha(w) dV_\alpha(z) < \infty.$$

Remark 5.10. Letting $q = 0$ in Theorems 5.8 and 5.9, we get [9, Theorem 1, Theorem 2] respectively as particular cases. Moreover, from the above theorem, it is clear that under the assumption of Theorem 5.9, if $f \in \mathcal{N}(p, q, s)$, then

$$\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \int_{\mathbb{B}} \left\{ \frac{|f(z) - f(w)|^p}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} (1 - |w|^2)^{\frac{q}{2}} (1 - |z|^2)^{\frac{q}{2}} (1 - |\Phi_a(w)|^2)^{\frac{ns}{2}} (1 - |\Phi_a(z)|^2)^{\frac{ns}{2}} \right\} dV_\alpha(w) dV_\alpha(z) < \infty.$$

We also have the following description.

Theorem 5.11. *Suppose $f \in H(\mathbb{B})$, $p \geq 1$, $q > 0$, $s > \max\{0, 1 - \frac{q}{n}\}$ and $0 < r < 1$. Then the following statements are equivalent:*

- (1) $f \in \mathcal{N}(p, q, s)$;
- (2)

$$\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left(\frac{1}{V(D(z, r))} \int_{D(z, r)} |f(z) - f(w| (1 - |z|^2)^{\frac{q}{2p}} (1 - |w|^2)^{\frac{q}{2p}} (1 - |\Phi_a(z)|^2)^{\frac{ns}{2p}} (1 - |\Phi_a(w)|^2)^{\frac{ns}{2p}} dV(w) \right)^p d\lambda(z) < \infty;$$

- (3)

$$\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left(\sup_{w \in D(z, r)} |f(z) - f(w| (1 - |z|^2)^{\frac{q}{2p}} (1 - |w|^2)^{\frac{q}{2p}} (1 - |\Phi_a(z)|^2)^{\frac{ns}{2p}} (1 - |\Phi_a(w)|^2)^{\frac{ns}{2p}} \right)^p d\lambda(z) < \infty;$$

- (4) *There exists some c satisfying $1 < c < \frac{1}{r}$, such that*

$$\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left(\frac{1}{V(D(z, cr))} \int_{D(z, cr)} |f(z) - f(w| (1 - |z|^2)^{\frac{q}{2}} (1 - |w|^2)^{\frac{q}{2}} (1 - |\Phi_a(z)|^2)^{\frac{ns}{2}} (1 - |\Phi_a(w)|^2)^{\frac{ns}{2}} dV(w) \right) d\lambda(z) < \infty;$$

Proof. (3) \implies (2). This implication is obvious.

(2) \implies (1). When $z \in D(a, r)$, $a, z \in \mathbb{B}$, we have

$$(5.5) \quad (1 - |z|^2)^{n+1} \simeq (1 - |a|^2)^{n+1} \simeq |1 - \langle a, z \rangle|^{n+1} \simeq V(D(a, r))$$

(see, e.g., [10, 25]) as well as

$$(5.6) \quad |1 - \langle z, u \rangle| \simeq |1 - \langle a, u \rangle|, \quad \forall u \in \mathbb{B}.$$

(see, e.g., [25, (2.20)]).

By the inequality

$$(1 - |z|^2)|\nabla f(z)| \lesssim \frac{1}{V(D(z, r))} \int_{D(z, r)} |f(z) - f(w)| dV(w), \quad \forall z \in \mathbb{B},$$

(see, e.g., [9]), we have

$$\begin{aligned} & |\nabla f(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} \\ & \lesssim \left(\frac{1}{V(D(z, r))} \int_{D(z, r)} |f(z) - f(w)| (1 - |z|^2)^{\frac{q}{p}} (1 - |\Phi_a(z)|^2)^{\frac{ns}{p}} dV(w) \right)^p \\ & \simeq \left(\frac{1}{V(D(z, r))} \int_{D(z, r)} |f(z) - f(w)| (1 - |z|^2)^{\frac{q}{2p}} (1 - |w|^2)^{\frac{q}{2p}} \right. \\ & \quad \left. (1 - |\Phi_a(z)|^2)^{\frac{ns}{2p}} (1 - |\Phi_a(w)|^2)^{\frac{ns}{2p}} dV(w) \right)^p. \end{aligned}$$

Integrating with respect to z over \mathbb{B} on both sides and taking the supremum over a , we get

$$\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\nabla f(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z),$$

which implies $f \in \mathcal{N}(p, q, s)$.

(1) \implies (4). Indeed, for this assertion, we can show that for each $1 \leq c < \frac{1}{r}$, (4) is satisfied. Take and fix some $c \in (1, \frac{1}{r})$. From Lemma 5.7 and the fact that

$$|\tilde{\nabla} f(z)|^2 = (1 - |z|^2)(|\nabla f(z)|^2 - |Rf(z)|^2),$$

we have

$$\int_{D(z, cr)} |f(w)|^p dV(w) \lesssim \int_{D(z, cr)} (1 - |w|^2)^p |\nabla f(w)|^p dV(w) + |f(z)|^p.$$

Hence, we have

$$\begin{aligned}
& \int_{\mathbb{B}} \left(\frac{1}{V(D(z, cr))} \int_{D(z, cr)} |f(z) - f(w)|^p (1 - |z|^2)^{\frac{q}{2}} (1 - |w|^2)^{\frac{q}{2}} \right. \\
& \quad \left. (1 - |\Phi_a(z)|^2)^{\frac{ns}{2}} (1 - |\Phi_a(w)|^2)^{\frac{ns}{2}} dV(w) \right) d\lambda(z) \\
& \lesssim \int_{\mathbb{B}} \left(\frac{1}{V(D(z, cr))} \int_{D(z, cr)} (1 - |w|^2)^{p+q} |\nabla f(w)|^p (1 - |\Phi_a(w)|^2)^{ns} dV(w) \right) d\lambda(z) \\
& \lesssim \int_{\mathbb{B}} \int_{\mathbb{B}} \chi_{D(z, cr)}(w) |\nabla f(w)|^p (1 - |w|^2)^{p+q} (1 - |\Phi_a(w)|^2)^{ns} d\lambda(w) d\lambda(z) \\
& = \int_{\mathbb{B}} \chi_{D(w, cr)}(z) \left(\int_{\mathbb{B}} |\nabla f(w)|^p (1 - |w|^2)^{p+q} (1 - |\Phi_a(w)|^2)^{ns} d\lambda(w) \right) d\lambda(z) \\
& \lesssim \|f\|^p < \infty.
\end{aligned}$$

(4) \implies (3). Suppose there exists some c satisfying (4). For any $f \in H(\mathbb{B})$, by the subharmonicity and [25, Proposition 1.21 and Lemma 2.20], for any $z \in \mathbb{B}$ and $w \in D(z, r)$, *i.e.* $|\Phi_z(w)| < r$, we have

$$\begin{aligned}
& |f(z) - f(w)|^p = |(f \circ \Phi_z)(\Phi_z(w)) - (f \circ \Phi_z)(0)|^p \\
& \lesssim \int_{\{u \in \mathbb{B}: |u - \Phi_z(w)| < (c-1)r\}} |(f \circ \Phi_z)(u) - (f \circ \Phi_z)(0)|^p dV(u) \\
& \leq \int_{|u| \leq cr} |(f \circ \Phi_z)(u) - f(z)|^p dV(u) \\
& \quad (\text{since } |\Phi_z(w)| < r) \\
& = \int_{|\Phi_z(\zeta)| < cr} |f(\zeta) - f(z)|^p \frac{(1 - |z|^2)^{n+1}}{|1 - \langle z, \zeta \rangle|^{2(n+1)}} dV(\zeta) \\
& \quad (\text{change variables with } \zeta = \Phi_z(u)) \\
& \simeq \frac{1}{V(D(z, cr))} \int_{D(z, cr)} |f(\zeta) - f(z)|^p dV(\zeta).
\end{aligned}$$

Hence, by the above inequality, we have

$$\begin{aligned}
& \int_{\mathbb{B}} \left(\sup_{w \in D(z, r)} |f(z) - f(w)| (1 - |z|^2)^{\frac{q}{2p}} (1 - |w|^2)^{\frac{q}{2p}} \right. \\
& \quad \left. (1 - |\Phi_a(z)|^2)^{\frac{ns}{2p}} (1 - |\Phi_a(w)|^2)^{\frac{ns}{2p}} \right)^p d\lambda(z) \\
& \simeq \int_{\mathbb{B}} (1 - |z|^2)^{\frac{q}{p}} (1 - |\Phi_a(z)|^2)^{\frac{ns}{p}} \left(\sup_{w \in D(z, r)} |f(z) - f(w)|^p \right) d\lambda(z)
\end{aligned}$$

$$\begin{aligned}
&\lesssim \int_{\mathbb{B}} \frac{1}{V(D(z, cr))} \int_{D(z, cr)} |f(\zeta) - f(z)|^p (1 - |z|^2)^{\frac{q}{p}} (1 - |\Phi_a(z)|^2)^{\frac{ns}{p}} dV(\zeta) d\lambda(z) \\
&\simeq \int_{\mathbb{B}} \left(\frac{1}{V(D(z, cr))} \int_{D(z, cr)} |f(z) - f(\zeta)|^p (1 - |z|^2)^{\frac{q}{2}} (1 - |\zeta|^2)^{\frac{q}{2}} \right. \\
&\quad \left. (1 - |\Phi_a(z)|^2)^{\frac{ns}{2}} (1 - |\Phi_a(\zeta)|^2)^{\frac{ns}{2}} dV(\zeta) \right) d\lambda(z) \\
&< \infty,
\end{aligned}$$

which implies the desired result. \square

From the Theorem 5.11, (5.5) and (5.6), we easily get the following result.

Theorem 5.12. *Suppose $f \in H(\mathbb{B})$, $p \geq 1$, $q > 0$, $s > \max\{0, 1 - \frac{q}{n}\}$ and $0 < r < 1$. Then the following statements are equivalent:*

- (1) $f \in \mathcal{N}(p, q, s)$;
- (2)

$$\begin{aligned}
&\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left(\frac{1}{V(D(z, r))} \int_{D(z, r)} |f(z) - f(w)| (1 - |z|^2)^{\frac{q}{p}} \right. \\
&\quad \left. (1 - |\Phi_a(z)|^2)^{\frac{ns}{p}} dV(w) \right)^p d\lambda(z) < \infty;
\end{aligned}$$

(3)

$$\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left(\sup_{w \in D(z, r)} |f(z) - f(w)| (1 - |z|^2)^{\frac{q}{p}} (1 - |\Phi_a(z)|^2)^{\frac{ns}{p}} \right)^p d\lambda(z) < \infty;$$

- (4) *There exists some c satisfying $1 < c < \frac{1}{r}$, such that*

$$\begin{aligned}
&\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left(\frac{1}{V(D(z, cr))} \int_{D(z, cr)} |f(z) - f(w)|^p (1 - |z|^2)^q \right. \\
&\quad \left. (1 - |\Phi_a(z)|^2)^{ns} \right) d\lambda(z) < \infty;
\end{aligned}$$

Remark 5.13. Our earlier estimates in Theorem 5.8, Theorem 5.9, Theorem 5.11 and Theorem 5.12 are pointwise estimates with respect to $a \in B$, so if we replace $\sup_{a \in \mathbb{B}}$ and $< \infty$ by $\lim_{|a| \rightarrow 1}$ and $= 0$, respectively, we obtain the corresponding characterizations of $\mathcal{N}^0(p, q, s)$. Hence, we omit the details of the proof.

6. ATOMIC DECOMPOSTION FOR $\mathcal{N}(p, q, s)$ -TYPE SPACES

In this section, we will focus on the decomposition of functions in $\mathcal{N}(p, q, s)$ -type spaces, which is an important concept and is a useful tool in studying such kind of function spaces. Before we establish the main results, we give some preliminaries.

Recall that for $z, w \in \mathbb{B}$, the Bergman metric can be written as

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\Phi_z(w)|}{1 - |\Phi_z(w)|}.$$

(see, e.g. [25, Proposition 1.21]). Moreover, for $r > 0$ and $z \in \mathbb{B}$, the set

$$E(z, r) = \{w \in \mathbb{B} : \beta(z, w) < r\}$$

is a Bergman metric ball at z . Note that by a simple calculation, we have

$$|\Phi_z(w)| = \tanh \beta(z, w), \quad z, w \in \mathbb{B},$$

which implies that $E(z, r) = D(z, \tanh r)$.

Lemma 6.1. [25, Theorem 2.23] *There exists a positive integer N such that for any $0 < r \leq 1$ we can find a sequence $\{a_k\}$ in \mathbb{B} with the following properties:*

- (1) $\mathbb{B} = \bigcup_k E(a_k, r)$;
- (2) The sets $E(a_k, r/4)$ are mutually disjoint;
- (3) Each point $z \in \mathbb{B}$ belongs to at most N of the sets $E(a_k, 4r)$.

Lemma 6.2. [25, Lemma 2.28] *Take and fix a sequence $\{a_k\}$ chosen according to Lemma 6.1 with r the separation constant. Then for each $k \geq 1$ there exists a Borel set E_k satisfying the following conditions:*

- (1) $E(a_k, r/4) \subset E_k \subset E(a_k, r)$ for every k ;
- (2) $E_k \cap E_j = \emptyset$ for $k \neq j$;
- (3) $\mathbb{B} = \bigcup_k E_k$.

Lemma 6.3. *Suppose $p \geq 1, q > 0, s > \max\{1, 1 - \frac{q}{n}\}$ and $\{a_k\} \subset \mathbb{B}$ is a chosen sequence according to Lemma 6.1 with the separation constant $r \in (0, 1]$. Then*

$$d\mu_1 = \sum_k |c_k|^p (1 - |a_k|^2)^{q+ns} \delta_{a_k} dV(z)$$

is an (ns) -Carleson measure if and only if

$$d\mu_2 = \sum_k \frac{|c_k|^p}{(1 - |a_k|^2)^p} (1 - |z|^2)^{p+q+ns} \chi_k(z) d\lambda(z)$$

is an (ns) -Carleson measure, where χ_k is the characteristic function of $E(a_k, r)$.

Proof. For any $a \in \mathbb{B}$, by [25, Lemma 2.20, (2.20)] and [18, 2.2.7], we have

$$\begin{aligned}
& \int_{\mathbb{B}} \left(\frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2} \right)^{ns} d\mu_2(z) \\
&= \int_{\mathbb{B}} \left(\frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2} \right)^{ns} \left(\sum_k \frac{|c_k|^p}{(1 - |a_k|^2)^p} (1 - |z|^2)^{p+q+ns} \chi_k(z) \right) d\lambda(z) \\
&= \sum_k \frac{|c_k|^p}{(1 - |a_k|^2)^p} \int_{D(a_k, r)} \frac{(1 - |a|^2)^{ns} (1 - |z|^2)^{p+q+ns-n-1}}{|1 - \langle z, a \rangle|^{2ns}} dV(z) \\
&\simeq \sum_k \frac{|c_k|^p}{(1 - |a_k|^2)^{n+1-q+ns}} \cdot \frac{(1 - |a|^2)^{ns}}{|1 - \langle a_k, a \rangle|^{2ns}} V(D(a_k, r)) \\
&\simeq \sum_k |c_k|^p (1 - |a_k|^2)^{q+ns} \cdot \frac{(1 - |a|^2)^{ns}}{|1 - \langle a_k, a \rangle|^{2ns}} \\
&= \int_{\mathbb{B}} \left(\frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2} \right)^{ns} d\mu_1(z).
\end{aligned}$$

The desired result follows from [27, Theorem 45]. \square

Fix a parameter $b > n$ and let $\alpha = b - (n + 1)$. We also fix a sequence $\{a_k\}$ chosen according to Lemma 6.1 with separation constant r and a sequence of Borel measurable sets $\{E_k\}$ with each E_k satisfying the condition in Lemma 6.2. Recall that the operator T associated to $\{a_k\}$ is as follows:

$$(6.1) \quad T(f)(z) = \int_{\mathbb{B}} \frac{(1 - |w|^2)^{b-n-1}}{|1 - \langle z, w \rangle|^b} f(w) dV(w),$$

where f is some Lebesgue measurable function.

Moreover, take a finer “lattice” $\{a_{kj}\}$ with separation constant γ in the Bergman metric than $\{a_k\}$ with a sequence of Borel measurable sets $\{E_{kj}\}$ chosen according to [25, Page 64] and define

$$(6.2) \quad S(f)(z) = \sum_{k,j} \frac{V_\alpha(E_{kj}) f(a_{kj})}{(1 - \langle z, a_{kj} \rangle)^b},$$

where $f \in H(\mathbb{B})$. Note that $\{a_{kj}\}$ also satisfies the conditions in Lemma 6.1. We refer the reader to the excellent book [25] for the detailed information about such a decomposition of \mathbb{B} into Bergman metric balls.

The following result indicates a deep relationship between T and S .

Lemma 6.4. [25, Lemma 3.22] *There exists a constant $C > 0$, independent of the separation constant r for $\{a_k\}$ and the separation constant*

γ for $\{a_{kj}\}$, such that

$$|f(z) - S(f)(z)| \leq C\sigma T(|f|)(z)$$

for all $f \in H(\mathbb{B})$ and $z \in \mathbb{B}_n$, where $\sigma = \gamma + \frac{\tanh(\gamma)}{\tanh(r)}$.

We have the following result considering the behaviour of T , which follows the methods in [25, Theorem 5.26] and [17, Lemma 1].

Lemma 6.5. *Let f be some Lebesgue measurable function. Suppose $p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{n}\}$ and $t > n - p - q - ns$. Then we have*

- (1) *If $p > 1, b > \frac{n+1}{p'} + \frac{q+ns+t}{p} + 1$ and*

$$|f(z)|^p (1 - |z|^2)^{p+q+ns+t} d\lambda(z)$$

is an (ns) -Carleson measure, where p' is the conjugate of p , then

$$|T(f)(z)|^p (1 - |z|^2)^{p+q+ns+t} d\lambda(z)$$

is also an (ns) -Carleson measure.

- (2) *If $p = 1, b > 1 + q + ns + t$ and*

$$|f(z)|(1 - |z|^2)^{1+q+ns+t} d\lambda(z)$$

is an (ns) -Carleson measure, then

$$|T(f)(z)|(1 - |z|^2)^{1+q+ns+t} d\lambda(z)$$

is also an (ns) -Carleson measure.

Proof. The proof of (2) is a simple modification of (1), which turns out to be much more easier than (1), and hence we omit it here.

For any $\xi \in \mathbb{S}$ and $\delta \in (0, 1)$, we have

$$\begin{aligned} & \frac{1}{\delta^{ns}} \int_{Q_\delta(\xi)} |T(f)(z)|^p (1 - |z|^2)^{p+q+ns+t} d\lambda(z) \\ & \leq \frac{1}{\delta^{ns}} \int_{Q_\delta(\xi)} \left(\int_{\mathbb{B}} \frac{(1 - |w|^2)^{b-n-1}}{|1 - \langle z, w \rangle|^b} |f(w)| dV(w) \right)^p (1 - |z|^2)^{p+q+ns+t} d\lambda(z) \\ & = \frac{1}{\delta^{ns}} \int_{Q_\delta(\xi)} \left(\left[\int_{Q_{2\delta}(\xi)} + \sum_{j=1}^{\infty} \int_{A_j} \right] \frac{(1 - |w|^2)^{b-n-1}}{|1 - \langle z, w \rangle|^b} |f(w)| dV(w) \right)^p (1 - |z|^2)^{p+q+ns+t} d\lambda(z) \\ & \leq \frac{2^{p-1}}{\delta^{ns}} \int_{Q_\delta(\xi)} \left(\int_{Q_{2\delta}(\xi)} \frac{(1 - |w|^2)^{b-n-1}}{|1 - \langle z, w \rangle|^b} |f(w)| dV(w) \right)^p (1 - |z|^2)^{p+q+ns+t} d\lambda(z) \\ & \quad + \frac{2^{p-1}}{\delta^{ns}} \int_{Q_\delta(\xi)} \left(\sum_{j=1}^{\infty} \int_{A_j} \frac{(1 - |w|^2)^{b-n-1}}{|1 - \langle z, w \rangle|^b} |f(w)| dV(w) \right)^p (1 - |z|^2)^{p+q+ns+t} d\lambda(z) \\ & = I_1 + I_2, \end{aligned}$$

where $A_j = \{w \in \mathbb{B} : 2^j \delta \leq |1 - \langle w, \xi \rangle| < 2^{j+1} \delta\}$, $i = 1, 2, \dots$

• **Estimation of I_1 .**

Consider the integral operator M induced by $K(z, w)$,

$$Mh(z) = \int_{\mathbb{B}} K(z, w)h(w)dV(w), z \in \mathbb{B},$$

where the kernel function

$$K(z, w) = \frac{(1 - |w|^2)^{b - \frac{n+1}{p'} - \frac{q+ns+t}{p} - 1} (1 - |z|^2)^{1 + \frac{q+ns+t-n-1}{p}}}{|1 - \langle z, w \rangle|^b}, \quad z, w \in \mathbb{B}.$$

We claim that M is a bounded operator on $L^p(\mathbb{B}, dV)$. Indeed, consider the function $g(z) = (1 - |z|^2)^{-\frac{1}{p+p'}}$. On one hand, we have

$$\begin{aligned} & \int_{\mathbb{B}} K(z, w)g^{p'}(w)dV(w) \\ &= (1 - |z|^2)^{1 + \frac{q+ns+t-n-1}{p}} \int_{\mathbb{B}} \frac{(1 - |w|^2)^{b - \frac{n+1}{p'} - \frac{q+ns+t}{p} - 1 - \frac{p'}{p+p'}}}{|1 - \langle z, w \rangle|^b} dV(w) \\ &= (1 - |z|^2)^{1 + \frac{q+ns+t-n-1}{p}} \int_{\mathbb{B}} \frac{(1 - |w|^2)^{b - \frac{n+1}{p'} - \frac{q+ns+t}{p} - 1 - \frac{p'}{p+p'}}}{|1 - \langle z, w \rangle|^{n+1 + (b - \frac{n+1}{p'} - \frac{q+ns+t}{p} - 1 - \frac{p'}{p+p'}) + c_1}} dV(w), \end{aligned}$$

where $c_1 = 1 + \frac{q+ns+t-n-1}{p} + \frac{p'}{p+p'}$. Note that by our assumption,

$$b - \frac{n+1}{p'} - \frac{q+ns+t}{p} - 1 - \frac{p'}{p+p'} > -1$$

and

$$c_1 = 1 + \frac{q+ns+t-n-1}{p} + \frac{p'}{p+p'} > 0,$$

and hence by [18, Proposition 1.4.10],

$$\begin{aligned} & \int_{\mathbb{B}} K(z, w)g^{p'}(w)dV(w) \\ & \lesssim (1 - |z|^2)^{1 + \frac{q+ns+t-n-1}{p}} \cdot (1 - |z|^2)^{-1 - \frac{q+ns+t-n-1}{p} - \frac{p'}{p+p'}} = g^{p'}(z). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \int_{\mathbb{B}} K(z, w)g^p(z)dV(z) \\ &= (1 - |w|^2)^{b - \frac{n+1}{p'} - \frac{q+ns+t}{p} - 1} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{1 + \frac{q+ns+t-n-1}{p} - \frac{p}{p+p'}}}{|1 - \langle z, w \rangle|^b} dV(z) \\ &= (1 - |w|^2)^{b - \frac{n+1}{p'} - \frac{q+ns+t}{p} - 1} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{1 + \frac{q+ns+t-n-1}{p} - \frac{p}{p+p'}}}{|1 - \langle z, w \rangle|^{n+1 + (1 + \frac{q+ns+t-n-1}{p} - \frac{p}{p+p'}) + c_2}} dV(z), \end{aligned}$$

where $c_2 = b + \frac{p}{p+p'} - \frac{n+1}{p'} - 1 - \frac{q+ns+t}{p}$. By our assumption again, it follows that

$$1 + \frac{q + ns + t - n - 1}{p} - \frac{p}{p + p'} > -1$$

and

$$c_2 = b + \frac{p}{p + p'} - \frac{n + 1}{p'} - 1 - \frac{q + ns + t}{p} > 0.$$

Thus, we have

$$\begin{aligned} & \int_{\mathbb{B}} K(z, w) g^p(z) dV(z) \\ & \lesssim (1 - |w|^2)^{b - \frac{n+1}{p'} - \frac{q+ns+t}{p} - 1} \cdot (1 - |w|^2)^{-b - \frac{p}{p+p'} + \frac{n+1}{p'} + 1 + \frac{q+ns+t}{p}} = g^p(w). \end{aligned}$$

The boundedness of M on $L^p(\mathbb{B}, dV)$ is then clear by Schur's test (see, e.g., [25, Theorem 2.9]). Put

$$h(w) = |f(w)|(1 - |w|^2)^{1 + \frac{q+ns+t-n-1}{p}} \chi_{Q_{2\delta}(\xi)}(w), \quad w \in \mathbb{B}.$$

It is easy to see $h \in L^p(\mathbb{B}, dV)$ since $|f(z)|^p(1 - |z|^2)^{p+q+ns+t} d\lambda(z)$ is an (ns) -Carleson measure. Moreover, we have

$$\frac{1}{(2\delta)^{ns}} \|h\|_{L^p}^p = \frac{1}{(2\delta)^{ns}} \int_{Q_{2\delta}(\xi)} |f(w)|^p (1 - |w|^2)^{p+q+ns+t} d\lambda(w) < \infty.$$

Therefore,

$$\begin{aligned} I_1 & \leq \frac{2^{p-1}}{\delta^{ns}} \int_{\mathbb{B}} \left(\int_{\mathbb{B}} K(z, w) h(w) dV(w) \right)^p dV(z) \\ & = \frac{2^{p-1}}{\delta^{ns}} \|Mh\|_{L^p}^p \lesssim \frac{1}{\delta^{ns}} \|h\|_{L^p}^p < \infty. \end{aligned}$$

• **Estimation of I_2 .**

Note that for $z \in Q_\delta(\xi)$ and $w \in A_j$, we have

$$|1 - \langle z, w \rangle|^{\frac{1}{2}} \geq |1 - \langle \xi, w \rangle|^{\frac{1}{2}} - |1 - \langle \xi, z \rangle|^{\frac{1}{2}} > \frac{1}{2}(\sqrt{2} - 1)2^{\frac{j}{2}}\delta^{\frac{1}{2}}.$$

Moreover, for each $j \geq 1$, consider the term

$$I_{3,j} = \int_{Q_{2^{j+1}\delta}(\xi)} |f(w)|(1 - |w|^2)^{b-n-1} dV(w).$$

Using Hölder's inequality and [25, Corollary 5.24], we have

$$\begin{aligned}
I_{3,j} &\leq \left(\int_{Q_{2^{j+1}\delta}(\xi)} |f(w)|^p (1 - |w|^2)^{p+q+ns+t} d\lambda(z) \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_{Q_{2^{j+1}\delta}(\xi)} (1 - |w|^2)^{p'(b-n-1)-p'(1+\frac{q+ns+t-n-1}{p})} dV(w) \right)^{\frac{1}{p'}} \\
&\lesssim (2^{j+1}\delta)^{b-1-\frac{q+ns+t}{p}} (2^{j+1}\delta)^{\frac{ns}{p}} \\
&\quad \times \left(\frac{1}{(2^{j+1}\delta)^{ns}} \int_{Q_{2^{j+1}\delta}(\xi)} |f(w)|^p (1 - |w|^2)^{p+q+ns+t} d\lambda(z) \right)^{\frac{1}{p}} \\
&\lesssim (2^{j+1}\delta)^{b-1-\frac{q+t}{p}}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
I_2 &\lesssim \frac{1}{\delta^{ns}} \int_{Q_\delta(\xi)} \left(\sum_{j=1}^{\infty} \frac{1}{(2^j\delta)^b} \int_{Q_{2^{j+1}\delta}(\xi)} |f(w)|(1 - |w|^2)^{b-n-1} dV(w) \right)^p \\
&\quad (1 - |z|^2)^{p+q+ns+t-n-1} dV(z) \\
&\simeq \delta^{p+q+t} \left(\sum_{j=1}^{\infty} \frac{1}{(2^j\delta)^b} \int_{Q_{2^{j+1}\delta}(\xi)} |f(w)|(1 - |w|^2)^{b-n-1} dV(w) \right)^p \\
&\lesssim \delta^{p+q+t} \left(\sum_{j=1}^{\infty} \frac{1}{(2^j\delta)^b} (2^{j+1}\delta)^{b-1-\frac{q+t}{p}} \right)^p \lesssim \left(\sum_{j=1}^{\infty} \frac{1}{2^{j(1+\frac{q+t}{p})}} \right)^p < \infty.
\end{aligned}$$

Finally, combining the estimations on I_1 and I_2 , it is clear that for any $\xi \in \mathbb{S}$ and $\delta \in (0, 1)$,

$$\frac{1}{\delta^{ns}} \int_{Q_\delta(\xi)} |T(f)(z)|^p (1 - |z|^2)^{p+q+ns+t} d\lambda(z) < \infty,$$

which implies $|T(f)(z)|^p (1 - |z|^2)^{p+q+ns+t} d\lambda(z)$ is an (ns) -Carleson measure. \square

We are now ready to establish our main results in this section.

Theorem 6.6. *Suppose $p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{n}\}$ and*

$$b > \begin{cases} \frac{n+1}{p'} + \frac{q+ns}{p}, & p > 1; \\ q + ns, & p = 1. \end{cases}$$

- (1) *Let $\{a_k\}$ be a sequence satisfying the conditions in Lemma 6.1 with the separation constant $r \in (0, 1)$. If $\{c_k\}$ is a sequence*

such that the measure $\sum_k |c_k|^p (1 - |a_k|^2)^{q+ns} \delta_{a_k}$ is an (ns) -Carleson measure, then the function

$$f(z) = \sum_k c_k \left(\frac{1 - |a_k|^2}{1 - \langle z, a_k \rangle} \right)^b$$

belongs to $\mathcal{N}(p, q, s)$.

- (2) There exists a sequence $\{a_k\}$ in \mathbb{B} such that $\mathcal{N}(p, q, s)$ -type spaces consists exactly of function of the form

$$f(z) = \sum_k c_k \left(\frac{1 - |a_k|^2}{1 - \langle z, a_k \rangle} \right)^b,$$

where the sequence $\{c_k\}$ has the property that $\sum_k |c_k|^p (1 - |a_k|^2)^{q+ns} \delta_{a_k}$ is an (ns) -Carleson measure.

Proof. Without the loss of generality, we may assume $p > 1$.

- (1). For each $k \geq 1$, let $E_k = E(a_k, \frac{r}{4})$. Consider the function

$$u(z) = \sum_{k=1}^{\infty} \frac{|c_k| \chi_k(z)}{1 - |a_k|^2},$$

where χ_k is the characteristic function of the set E_k . By Lemma 6.1, the sets E_k are mutually disjoint, the measure $|u(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z)$ can be written as

$$\begin{aligned} & \left| \sum_{k=1}^{\infty} \frac{|c_k| \chi_k(z)}{1 - |a_k|^2} \right|^p (1 - |z|^2)^{p+q+ns} d\lambda(z) \\ &= \sum_{k=1}^{\infty} \frac{|c_k|^p}{(1 - |a_k|^2)^p} (1 - |z|^2)^{p+q+ns} \chi_k(z) d\lambda(z), \end{aligned}$$

which by Lemma 6.3, is an (ns) -Carleson measure.

Let T be the operator with the parameter $b+1$. By Lemma 6.5, since $|u(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z)$ is an (ns) -Carleson measure, it follows that $|T(u)(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z)$ is also an (ns) -Carleson measure.

From the proof of [25, Lemma 5.28], we know that $|Rf(z)| \lesssim T(u)(z)$, which implies that

$$|Rf(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z)$$

is also an (ns) -Carleson measure. The desired result follows from Theorem 5.3.

- (2). Let X be the function space consist $f \in H(\mathbb{B})$ satisfying

$$\|f\|_X^p = \sup_{0 < \delta < 1, \xi \in \mathbb{S}} \frac{1}{\delta^{ns}} \int_{Q_\delta(\xi)} |f(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z) < \infty.$$

It is easy to check that X becomes a Banach space when equipped with the norm $\|\cdot\|_X$ (see, e.g., [25, Page 185]). Hence, by Lemma 6.5, the operator T defined in (6.1) with parameter $b+1$ is bounded on X .

Take and fix a sequence $\{b_k\}$ satisfying the condition in Lemma 6.1 with separation constant r and a finer “lattice” $\{b_{kj}\}$ with separation constant γ satisfying that

$$C\sigma\|T\| < 1,$$

where C and σ are defined in Lemma 6.4. Let S be the linear operator defined in (6.2) with the parameter $b+1$ associated to $\{b_{kj}\}$. Thus, by Lemma 6.4 again, we can get

$$\|f - Sf\|_X \leq C\sigma\|T(\|f\|)\|_X \leq C\sigma\|T\|\|f\|_X < \|f\|_X,$$

which implies the operator $I - S$ is bounded on X with operator norm strictly less than 1, where I is the identity operator. Hence, by [24, Theorem 1.5.2], S is invertible on X .

Fix $f \in \mathcal{N}(p, q, s)$ and let $g = R^{\alpha,1}f$, where $\alpha = b - (n+1)$ and $R^{\alpha,1}$ is a linear partial differential operator of order 1 (see, e.g., [25, Proposition 1.15]). By [25, Proposition 1.15] and Theorem 5.3, $g \in X$.

Since S is invertible on X , there exists a function $h \in X$ such that $g = Sh$. Thus g admits the representation

$$g(z) = \sum_{k,j} \frac{V_\beta(E_{kj})h(b_{kj})}{(1 - \langle z, b_{kj} \rangle)^{b+1}}.$$

where $\beta = (b+1) - (n+1) = b - n$. Applying the inverse of $R^{\alpha,1}$ to both side with [25, Proposition 1.14], we obtain

$$f(z) = \sum_{k,j} \frac{V_\beta(E_{kj})h(b_{kj})}{(1 - \langle z, b_{kj} \rangle)^b}.$$

Let

$$c_{kj} = \frac{V_\beta(E_{kj})h(b_{kj})}{(1 - |b_{kj}|^2)^b}, \quad k \geq 1, 1 \leq j \leq J,$$

where J is some integer depending on γ (see, e.g., [25, Page 64]) and hence we can write

$$f(z) = \sum_{k,j} c_{kj} \left(\frac{1 - |b_{kj}|^2}{1 - \langle z, b_{kj} \rangle} \right)^b.$$

It remains for us to show that the measure

$$\sum_{k,j} |c_{kj}|^p (1 - |b_{kj}|^2)^{q+ns} \delta_{b_{kj}}$$

is an (ns) -Carleson measure. Since

$V_\beta(E_{kj}) \leq V_\beta(E_k) \simeq (1 - |b_k|^2)^{n+1+\beta} = (1 - |b_k|^2)^{b+1} \simeq (1 - |b_{kj}|^2)^{b+1}$, where the last estimation follows from the fact that $b_{kj} \in E(b_k, r)$, $1 \leq j \leq J$, it suffices to show that the measure

$$d\mu = \sum_{k,j} (1 - |b_{kj}|^2)^{p+q+ns} |h(b_{kj})|^p \delta_{b_{kj}}$$

is an (ns) -Carleson measure.

By Lemma 6.2, we know that $E(b_{kj}, \frac{\gamma}{4})$ are mutually disjoint. Using [25, Lemma 2.20, (2.20) and Lemma 2.24], we have for any $a \in \mathbb{B}$,

$$\begin{aligned} & \int_{\mathbb{B}} \left(\frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2} \right)^{ns} d\mu(z) \\ &= \sum_{k,j} \left(\frac{1 - |a|^2}{|1 - \langle b_{kj}, a \rangle|^2} \right)^{ns} |h(b_{kj})|^p (1 - |b_{kj}|^2)^{p+q+ns} \\ &\lesssim \sum_{k,j} \left(\frac{1 - |a|^2}{|1 - \langle b_{kj}, a \rangle|^2} \right)^{ns} \int_{E(b_{kj}, \frac{\gamma}{4})} |h(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z) \\ &\simeq \sum_{k,j} \int_{E(b_{kj}, \frac{\gamma}{4})} \left(\frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2} \right)^{ns} |h(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z) \\ &\leq \int_{\mathbb{B}} \left(\frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2} \right)^{ns} |h(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z) \\ &< \infty, \end{aligned}$$

where the last inequality follows from the fact that $f \in X$ and [27, Theorem 45]. Using [27, Theorem 45] again, we conclude that the measure μ is an (ns) -Carleson measure. The proof is complete. \square

Similarly, we have the following description for $\mathcal{N}^0(p, q, s)$ with regarding its atomic decomposition. First we observe that we have the following “little” version for Lemma 6.5.

Lemma 6.7. *Let f be some Lebesgue measurable function. Suppose $p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{n}\}$ and $t > n - p - q - ns$. Then we have*

- (1) *If $p > 1, b > \frac{n+1}{p'} + \frac{q+ns+t}{p} + 1$ and*

$$|f(z)|^p (1 - |z|^2)^{p+q+ns+t} d\lambda(z)$$

is a vanishing (ns) -Carleson measure, where p' is the conjugate of p , then

$$|T(f)(z)|^p (1 - |z|^2)^{p+q+ns+t} d\lambda(z)$$

is also a vanishing (ns) -Carleson measure.

(2) If $p = 1, b > 1 + q + ns + t$ and

$$|f(z)|(1 - |z|^2)^{1+q+ns+t}d\lambda(z)$$

is a vanishing (ns) -Carleson measure, then

$$|T(f)(z)|(1 - |z|^2)^{1+q+ns+t}d\lambda(z)$$

is also a vanishing (ns) -Carleson measure.

Proof. We would only consider the case $p > 1$ again. By our assumption, for any $\varepsilon > 0$, there exists a $\delta_0 > 0$, such that the estimate

$$(6.3) \quad \frac{1}{\delta^{ns}} \int_{Q_\delta(\xi)} |f(z)|^p (1 - |z|^2)^{p+q+ns+t} d\lambda(z) < \varepsilon$$

holds uniformly for $\xi \in \mathbb{S}$ when $\delta < \delta_0$. From the proof of Lemma 6.5, we have

$$I_1 \lesssim \frac{1}{\delta^{ns}} \int_{Q_{2\delta}(\xi)} |f(z)|^p (1 - |z|^2)^{p+q+ns+t} d\lambda(z),$$

which, combining with (6.3), implies when $\delta < \frac{\delta_0}{2}$, we have

$$I_1 \lesssim 2^{ns} \varepsilon.$$

Now we estimate I_2 . For the chosen ε , there exists a $J_0 \in \mathbb{N}$, such that

$$\sum_{j=J_0+1}^{\infty} \frac{1}{2^{j(1+\frac{q+1}{p})}} < \varepsilon^{1/p}.$$

From the proof of Lemma 6.5, we have

$$\begin{aligned} I_2 &\lesssim \frac{1}{\delta^{ns}} \int_{Q_\delta(\xi)} \left(\sum_{j=1}^{\infty} \frac{1}{(2^j \delta)^b} \int_{Q_{2^{j+1}\delta}(\xi)} |f(w)|(1 - |w|^2)^{b-n-1} dV(w) \right)^p \\ &\quad (1 - |z|^2)^{p+q+ns+t-n-1} dV(z) \\ &\lesssim \frac{1}{\delta^{ns}} \int_{Q_\delta(\xi)} \left(\sum_{j=1}^{J_0} \frac{1}{(2^j \delta)^b} \int_{Q_{2^{j+1}\delta}(\xi)} |f(w)|(1 - |w|^2)^{b-n-1} dV(w) \right)^p \\ &\quad (1 - |z|^2)^{p+q+ns+t-n-1} dV(z) \\ &\quad + \frac{1}{\delta^{ns}} \int_{Q_\delta(\xi)} \left(\sum_{j=J_0+1}^{\infty} \frac{1}{(2^j \delta)^b} \int_{Q_{2^{j+1}\delta}(\xi)} |f(w)|(1 - |w|^2)^{b-n-1} dV(w) \right)^p \\ &\quad (1 - |z|^2)^{p+q+ns+t-n-1} dV(z) \\ &= J_1 + J_2. \end{aligned}$$

• **Estimation of J_1 .**

Take $\delta < \frac{\delta_0}{2^{J_0+1}}$. Then by (6.3) and the estimation of $I_{3,j}$, $j \geq 1$, we have for $j = 1, 2, \dots, J_0$,

$$\begin{aligned} I_{3,j} &\lesssim (2^{j+1}\delta)^{b-1-\frac{q+t}{p}} \left(\frac{1}{(2^{j+1}\delta)^{ns}} \int_{Q_{2^{j+1}\delta}(\xi)} |f(w)|^p (1-|w|^2)^{p+q+ns+t} d\lambda(z) \right)^{\frac{1}{p}} \\ &\leq (2^{j+1}\delta)^{b-1-\frac{q+t}{p}} \varepsilon^{\frac{1}{p}}. \end{aligned}$$

Thus, we see that

$$\begin{aligned} J_1 &\lesssim \frac{\varepsilon}{\delta^{ns}} \int_{Q_\delta(\xi)} \left(\sum_{j=1}^{J_0} \frac{1}{(2^j\delta)^{1+\frac{q+t}{p}}} \right)^p (1-|z|^2)^{p+q+ns+t-n-1} dV(z) \\ &\lesssim \varepsilon \cdot \left(\sum_{j=1}^{J_0} \frac{1}{2^{j(1+\frac{q+t}{p})}} \right)^p \lesssim \varepsilon. \end{aligned}$$

• **Estimation of J_2 .**

By the proof in Lemma 6.5, it is easy to see that

$$J_2 \lesssim \left(\sum_{j=J_0+1}^{\infty} \frac{1}{2^{j(1+\frac{q+t}{p})}} \right)^p < \varepsilon.$$

Hence, it follows that $I_2 \lesssim \varepsilon$, which implies the desired result. \square

By using the last lemma, we get the following theorem considering the atomic decomposition on $\mathcal{N}^0(p, q, s)$ -type space, whose proof is straightforward from Theorem 6.6, and hence we omit the detail here.

Theorem 6.8. *Suppose $p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{n}\}$ and*

$$b > \begin{cases} \frac{n+1}{p'} + \frac{q+ns}{p}, & p > 1; \\ q + ns, & p = 1. \end{cases}$$

- (1) *Let $\{a_k\}$ be a sequence satisfying the conditions in Lemma 6.1 with the separation constant $r \in (0, 1)$. If $\{c_k\}$ is a sequence such that the measure $\sum_k |c_k|^p (1-|a_k|^2)^{q+ns} \delta_{a_k}$ is a vanishing (ns) -Carleson measure, then the function*

$$f(z) = \sum_k c_k \left(\frac{1-|a_k|^2}{1-\langle z, a_k \rangle} \right)^b$$

belongs to $\mathcal{N}^0(p, q, s)$.

- (2) *There exists a sequence $\{a_k\}$ in \mathbb{B} such that $\mathcal{N}^0(p, q, s)$ -type spaces consists exactly of function of the form*

$$f(z) = \sum_k c_k \left(\frac{1 - |a_k|^2}{1 - \langle z, a_k \rangle} \right)^b,$$

where the sequence $\{c_k\}$ has the property that $\sum_k |c_k|^p (1 - |a_k|^2)^{q+ns} \delta_{a_k}$ is a vanishing (ns) -Carleson measure.

We already see that the atomic decomposition on $\mathcal{N}(p, q, s)$ -type space is closely related to the choice of $\{a_k\}$. We say that a sequence $\{a_k\}$ of distinct points with in \mathbb{B} is an r -lattice in the Bergman metric if it satisfies Lemma 6.1 with separation constant r . We need the following lemma.

Lemma 6.9. *Let $\{z_n\}$ be a sequence of points on \mathbb{B} , $\alpha > -1$ and $f \in A_\alpha^1(\mathbb{B})$. Let $\{a_n\}$ is an r -lattice in the Bergman metric, then there exists a constant $C_1 > 0$ depending only on r and α so that*

$$(6.4) \quad \|f\|_{1,\alpha} \geq C_1 \sum_n (1 - |a_n|^2)^{n+1+\alpha} |f(a_n)|.$$

Furthermore, there exists some $r_0 > 0$ and a constant $C_2 > 0$ depending only on r and α so that

$$(6.5) \quad \|f\|_{1,\alpha} \leq C_2 \sum_n (1 - |a_n|^2)^{n+1+\alpha} |f(a_n)|,$$

if $0 < r < r_0$.

Proof. The first part of the above lemma is a particular case of [8, Lemma 1.5] and the second part is proved in [13, Theorem 2]. \square

Using the above lemma, we have the following result.

Theorem 6.10. *Let $p \geq 1$, $q > 0$, $s > \max\{0, 1 - \frac{q}{n}\}$, $\alpha > -1$ and $\{a_n\}$ be an r -lattice in the Bergman metric in \mathbb{B} . Then for any $\{c_n\} \in \ell^\infty$,*

$$(6.6) \quad f(z) = \sum_n c_n \cdot \left(\frac{1 - |a_n|^2}{1 - \langle z, a_n \rangle} \right)^{n+1+\alpha}$$

belongs to $\mathcal{N}(p, q, s)$. Moreover, there is an $r_0 > 0$ such that every $f \in \mathcal{N}(p, q, s)$ has the form (6.6) for some $\{c_n\} \in \ell^\infty$ if $r < r_0$.

Proof. Let $\{a_n\}$ be an r -lattice in the Bergman metric in \mathbb{B} . Moreover, since $s > 1 - \frac{q}{n}$, there exists some $k_0 \in \left(0, \frac{q}{p}\right]$, such that

$$s > 1 - \frac{q - k_0 p}{n},$$

which, by Proposition 2.4, implies $A^{-k_0}(\mathbb{B}) \subseteq \mathcal{N}(p, q, s)$.

Then T , defined as follows, is a bounded linear operator from $A_\alpha^1(\mathbb{B})$ to ℓ^1 ,

$$Tf = \{(Tf)_n\} = \{(1 - |a_n|^2)^{n+1+\alpha} f(a_n)\}, \quad f \in A_\alpha^1(\mathbb{B}),$$

where the boundedness of T is due to (6.4) under $\{z_n\}$ being an r -lattice. Thus T^* , by [25, Theorem 7.6], the adjoint operator of T is a bounded linear operator from $\ell^\infty (= (\ell^1)^*)$ to $A^{-k_0} (= \mathcal{B}^{k_0+1} = (A_\alpha^1)^*)$, where \mathcal{B}^{k_0+1} is the $(k_0 + 1)$ -Bloch space. Moreover, T^* can be written as follows.

$$\langle Tf, y \rangle = \langle f, T^*y \rangle, \quad f \in A_\alpha^1(\mathbb{B}), \quad y \in \ell^\infty,$$

where the $\langle \cdot, \cdot \rangle$ is just the usual inner product between ℓ^1 and ℓ^∞ .

To compute T^* , we take

$$y = e_n, \quad (e_n)_m = \begin{cases} 1, & m = n \\ 0, & m \neq n, \end{cases}$$

so, for $f \in A_\alpha^1(\mathbb{B})$,

$$\begin{aligned} \langle Tf, e_n \rangle &= (Tf)_n = (1 - |a_n|^2)^{n+1+\alpha} f(a_n) \\ &= (1 - |a_n|^2)^{n+1+\alpha} \langle f, K_{a_n} \rangle_{\alpha+k_0}, \end{aligned}$$

where $K_{a_n}(z) = \frac{1}{(1 - \langle z, a_n \rangle)^{n+1+\alpha}}$ is the reproducing kernel for $A_\alpha^1(\mathbb{B})$ and $\langle \cdot, \cdot \rangle_{\alpha+k_0}$ is the integral pair defined in [25, Theorem 7.6]. Hence

$$T^*e_n = (1 - |z_n|^2)^{n+1+\alpha} K_{a_n}(z)$$

and

$$T^*y = \sum_n c_n \cdot \left(\frac{1 - |a_n|^2}{1 - \langle z, a_n \rangle} \right)^{n+1+\alpha}, \quad \text{for } y = \{c_n\} \in \ell^\infty,$$

i.e. the function in the form (6.6) is in A^{-k_0} . Since $A^{-k_0} \subseteq \mathcal{N}(p, q, s)$, which implies the desired result.

Now we turn to showing the second part. Noting that by [25, Theorem 7.6] again, $(A_\alpha^1)^* = \mathcal{B}^{\frac{q}{p}+1} = A^{-\frac{q}{p}}(\mathbb{B})$, we also can regard T^* as a bounded linear operator from ℓ^∞ to $A^{-\frac{q}{p}}(\mathbb{B})$, where in the sequel, we denote T^* as S^* .

In fact, it is only necessary to claim S^* to be surjective. However, S^* is onto if and only if $S : A_\alpha^1(\mathbb{B}) \mapsto \ell^1$ is bounded below (see, e.g., [2, Theorem A, Page 194]). By Lemma 6.9, there exists an $r_0 > 0$ such that S is bounded below if $\{a_n\}$ is an r -lattice with $0 < r < r_0$, that is to say, there is an $r_0 > 0$, such that every $f \in A^{-\frac{q}{p}}(\mathbb{B})$ has the

form 6.6 for some $\{c_n\} \in \ell^\infty$. Finally, by Proposition 2.1, we note that $\mathcal{N}(p, q, s) \subseteq A^{-\frac{q}{p}}(\mathbb{B})$. The proof is complete. \square

By modifying the proof of the above theorem a bit, we easily get the following corollary.

Corollary 6.11. *Let $p > 0$, $\alpha > -1$ and $\{a_n\}$ be an r -lattice in the Bergman metric in \mathbb{B} . Then for any $\{c_n\} \in \ell^\infty$,*

$$(6.7) \quad f(z) = \sum_n c_n \cdot \left(\frac{1 - |a_n|^2}{1 - \langle z, a_n \rangle} \right)^{n+1+\alpha}$$

belongs to $A^{-p}(\mathbb{B})$. Moreover, there is an $r_0 > 0$ such that every $f \in A^{-p}(\mathbb{B})$ has the form (6.7) for some $\{c_n\} \in \ell^\infty$ if $r < r_0$.

7. DISTANCE BETWEEN $A^{-\frac{q}{p}}(\mathbb{B})$ SPACES AND $\mathcal{N}(p, q, s)$ -TYPE SPACES

Recall that from Theorem 2.1, we have, for $p \geq 1$ and $q, s > 0$,

$$\mathcal{N}(p, q, s) \subseteq A^{-\frac{q}{p}}(\mathbb{B}).$$

A natural question can be asked is: for any $f \in A^{-\frac{q}{p}}(\mathbb{B})$, what can we say about the distance between f and $A^{-\frac{q}{p}}(\mathbb{B})$ with regarding $\mathcal{N}(p, q, s)$ as a subspace of $A^{-\frac{q}{p}}(\mathbb{B})$? In this section, we will focus on this question.

We denote the distance in $A^{-\frac{q}{p}}(\mathbb{B})$ of f to $\mathcal{N}(p, q, s)$ by $d(f, \mathcal{N}(p, q, s))$, that is

$$d(f, \mathcal{N}(p, q, s)) = \inf_{g \in \mathcal{N}(p, q, s)} |f - g|_{\frac{q}{p}},$$

where $|\cdot|_{\frac{q}{p}}$ is the norm defined on $A^{-\frac{q}{p}}(\mathbb{B})$. Moreover, for $f \in H(\mathbb{B})$ and $\varepsilon > 0$, let

$$\Omega_\varepsilon(f) = \{z \in \mathbb{B} : |f(z)|(1 - |z|^2)^{\frac{q}{p}} \geq \varepsilon\}.$$

We have the following result.

Theorem 7.1. *Suppose $p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{n}\}$ and $f \in A^{-\frac{q}{p}}(\mathbb{B})$. Then the following quantities are equivalent:*

- (1) $d_1 = d(f, \mathcal{N}(p, q, s))$;
- (2) $d_2 = \inf\{\varepsilon : \chi_{\Omega_\varepsilon(f)}(z)(1 - |z|^2)^{ns} d\lambda(z) \text{ is an } (ns)\text{-Carleson measure}\}$;
- (3)

$$d_3 = \inf \left\{ \varepsilon : \sup_{a \in \mathbb{B}} \int_{\Omega_\varepsilon(f)} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty \right\}.$$

Proof. (1). $d_1 \lesssim d_2$.

Without the loss of generality, we may assume that $p > 1$. Let ε be a positive number such that $\chi_{\Omega_\varepsilon(f)}(z)(1-|z|^2)^{ns}d\lambda(z)$ is an (ns) -Carleson measure. Since $f \in A^{-\frac{q}{p}}(\mathbb{B})$, we have

$$\sup_{z \in \mathbb{B}} |f(z)|(1-|z|^2)^{\frac{q}{p}} < \infty.$$

Take and fix some $\alpha > \max \left\{ \frac{q}{p} - 1, \frac{q+ns-n-1}{p} \right\}$. It is easy to see that $f \in A_\alpha^1$, and hence by [25, Theorem 2.2], we have

$$f(z) = \int_{\mathbb{B}} \frac{f(w)}{(1-\langle z, w \rangle)^{n+1+\alpha}} dV_\alpha(w).$$

Let

$$f_1(z) = \int_{\Omega_\varepsilon(f)} \frac{f(w)}{(1-\langle z, w \rangle)^{n+1+\alpha}} dV_\alpha(w)$$

and

$$f_2(z) = \int_{\mathbb{B} \setminus \Omega_\varepsilon(f)} \frac{f(w)}{(1-\langle z, w \rangle)^{n+1+\alpha}} dV_\alpha(w).$$

It is clear that both f_1 and f_2 are holomorphich functions in \mathbb{B} and $f(z) = f_1(z) + f_2(z)$. We have the following claims.

- $|f_2|_{\frac{q}{p}} \leq C\varepsilon$ **for some constant** $C > 0$.

For any $z \in \mathbb{B}$, by [18, Proposition 1.4.10], we have

$$\begin{aligned} |f_2(z)| &\leq \int_{\mathbb{B} \setminus \Omega_\varepsilon(f)} \frac{|f(w)|}{|1-\langle z, w \rangle|^{n+1+\alpha}} dV_\alpha(w) \\ &\lesssim \varepsilon \int_{\mathbb{B}} \frac{(1-|w|^2)^{\alpha-\frac{q}{p}}}{|1-\langle z, w \rangle|^{n+1+\alpha}} dV_\alpha(w) \\ &\lesssim \varepsilon (1-|z|^2)^{-\frac{q}{p}}, \end{aligned}$$

which implies the desired claim.

- $f_1 \in \mathcal{N}(p, q, s)$.

By Theorem 5.3, it suffices to show that $|f_1(z)|^p(1-|z|^2)^{q+ns}d\lambda(z)$ is a (ns) -Carleson measure. Note that for $z \in \mathbb{B}$, we have

$$\begin{aligned} |f_1(z)| &\lesssim \int_{\Omega_\varepsilon(f)} \frac{|f(w)|(1-|w|^2)^{\frac{q}{p}}(1-|w|^2)^{\alpha-\frac{q}{p}} dV(w)}{|1-\langle z, w \rangle|^{n+1+\alpha}} \\ &\lesssim \int_{\Omega_\varepsilon(f)} \frac{(1-|w|^2)^{\alpha-\frac{q}{p}} dV(w)}{|1-\langle z, w \rangle|^{n+1+\alpha}} \\ &= \int_{\mathbb{B}} \frac{(1-|w|^2)^\alpha}{|1-\langle z, w \rangle|^{n+1+\alpha}} \cdot \frac{\chi_{\Omega_\varepsilon(f)}(w)}{(1-|w|^2)^{\frac{q}{p}}} dV(w). \end{aligned}$$

Let

$$g(w) = \frac{\chi_{\Omega_\varepsilon(f)}(w)}{(1 - |w|^2)^{\frac{q}{p}}}$$

and hence we have

$$|g(w)|^p (1 - |w|^2)^{q+ns} d\lambda(w) = \chi_{\Omega_\varepsilon(f)}(w) (1 - |w|^2)^{ns} d\lambda(w),$$

which is an (ns) -Carleson measure by our assumption. Now in Theorem 6.5, let $t = -p$ and $b = n + 1 + \alpha$, it is easy to check that

$$t = -p > n - p - q - ns$$

and

$$b = n + 1 + \alpha > \frac{n + 1}{p'} + \frac{q + ns}{p}.$$

Hence, the operator T with parameter $n + 1 + \alpha$ sends the (ns) -Carleson measure

$$|g(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z)$$

to

$$\begin{aligned} & |Tg(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z) \\ &= \left| \int_{\mathbb{B}} \frac{(1 - |w|^2)^\alpha}{|1 - \langle z, w \rangle|^{n+1+\alpha}} \cdot \frac{\chi_{\Omega_\varepsilon(f)}(w)}{(1 - |w|^2)^{\frac{q}{p}}} dV(w) \right|^p (1 - |z|^2)^{q+ns} d\lambda(z), \end{aligned}$$

which, by Theorem 6.5, is also an (ns) -Carleson measure. Finally, since

$$|f_1(z)| \lesssim \int_{\mathbb{B}} \frac{(1 - |w|^2)^\alpha}{|1 - \langle z, w \rangle|^{n+1+\alpha}} \cdot \frac{\chi_{\Omega_\varepsilon(f)}(w)}{(1 - |w|^2)^{\frac{q}{p}}} dV(w),$$

it follows that $|f_1(z)|^p (1 - |z|^2)^{q+ns} d\lambda(z)$ is an (ns) -Carleson measure. Hence, the claim is proved.

Note that $f_1 \in A^{-\frac{q}{p}}(\mathbb{B})$ since $\mathcal{N}(p, q, s) \subseteq A^{-\frac{q}{p}}(\mathbb{B})$. Thus, we have

$$d_1 = d(f, \mathcal{N}(p, q, s)) \leq |f - f_1|_{\frac{q}{p}} = |f_2|_{\frac{q}{p}} \lesssim \varepsilon.$$

Finally, by letting ε tends to d_2 , we get the desired result.

(2). $d_2 \leq d_3$.

Take and fix an $\varepsilon > 0$ such that

$$\sup_{a \in \mathbb{B}} \int_{\Omega_\varepsilon(f)} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty.$$

Since $|f(z)|^p (1 - |z|^2)^q \geq \varepsilon^p$ for $z \in \Omega_\varepsilon(f)$, it follows that

$$\begin{aligned} & \sup_{a \in \mathbb{B}} \int_{\Omega_\varepsilon(f)} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\ &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left(\frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2} \right)^{ns} \chi_{\Omega_\varepsilon(f)}(z) (1 - |z|^2)^{ns} d\lambda(z) < \infty, \end{aligned}$$

which, by [25, Theorem 45], implies $\chi_{\Omega_\varepsilon(f)}(z)(1 - |z|^2)^{ns}d\lambda(z)$ is an (ns) -Carleson measure. Hence,

$$\begin{aligned} & \left\{ \varepsilon : \sup_{a \in \mathbb{B}} \int_{\Omega_\varepsilon(f)} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty \right\} \\ & \subseteq \left\{ \varepsilon : \chi_{\Omega_\varepsilon(f)}(z)(1 - |z|^2)^{ns}d\lambda(z) \text{ is an } (ns)\text{-Carleson measure} \right\}, \end{aligned}$$

which implies $d_2 \leq d_3$.

(3). $d_3 \leq d_1$.

It suffices to show that

$$(d_1, \infty) \subseteq \left\{ \varepsilon : \sup_{a \in \mathbb{B}} \int_{\Omega_\varepsilon(f)} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty \right\}.$$

Take and fix any $\varepsilon > d_1$, then there exists a $f_1 \in \mathcal{N}(p, q, s)$, such that

$$|f - f_1|_{\frac{q}{p}} < \frac{d_1 + \varepsilon}{2}.$$

By triangle inequality, we have for $z \in \Omega_\varepsilon(f)$,

$$\begin{aligned} |f_1(z)|(1 - |z|^2)^{\frac{q}{p}} & \geq |f(z)|(1 - |z|^2)^{\frac{q}{p}} - |f(z) - f_1(z)|(1 - |z|^2)^{\frac{q}{p}} \\ & \geq \varepsilon - \frac{d_1 + \varepsilon}{2} = \frac{\varepsilon - d_1}{2}. \end{aligned}$$

Thus, we have

$$\begin{aligned} & \sup_{a \in \mathbb{B}} \int_{\Omega_\varepsilon(f)} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\ & \lesssim \sup_{a \in \mathbb{B}} \int_{\Omega_\varepsilon(f)} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\ & \leq \frac{2^p}{(\varepsilon - d_1)^p} \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f_1(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\ & < \infty, \end{aligned}$$

which implies the desired inclusion. The proof is complete. \square

Noting that by Corollary 5.5, we can express the $\mathcal{N}(p, q, s)$ -norm by using complex gradient and radial derivative respectively. By using these equivalent norms, we can express $d(f, \mathcal{N}(p, q, s))$ via different forms.

Let $\alpha > 0$. The α -Bloch space \mathcal{B}^α is the space of all holomorphic functions f on \mathbb{B} such that $\sup_{a \in \mathbb{B}} (1 - |z|^2)^\alpha |Rf(z)| < \infty$ and the little α -Bloch space \mathcal{B}_0^α consists of those holomorphic functions on \mathbb{B} satisfying $\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |Rf(z)| = 0$. It is well-known that \mathcal{B}^α becomes a

Banach space with the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z \in \mathbb{B}} (1 - |z|^2)^\alpha |Rf(z)|$. (see, e.g., [10]). Moreover, denote

$$J_1 = |f(0)| + \sup_{z \in \mathbb{B}} (1 - |z|^2)^\alpha |\nabla f(z)|$$

and

$$J_2 = |f(0)| + \sup_{z \in \mathbb{B}} (1 - |z|^2)^{\alpha-1} |\tilde{\nabla} f(z)|,$$

it is known that when $\alpha > 0$, J_1 and $\|f\|_{\mathcal{B}^\alpha}$ are equivalent and when $\alpha > \frac{1}{2}$, the same is true for both J_1 and J_2 (see, e.g., [25, Theorem 7.1 and Theorem 7.2]).

The α -Bloch space \mathcal{B}^α has a close relationship with the Bergman-type space $A^{-p}(\mathbb{B})$. It is a classical result that for $p > 1$, we have

$$A^{-p}(\mathbb{B}) = \mathcal{B}^{p+1}.$$

The proof for the above result in unit disc case is established in [25]. However, we did not find a reference for the unit ball case, and hence we give its proof here for the readers' convenience.

Lemma 7.2. *Suppose $p > 0$. Then f is in \mathcal{B}^{p+1} if and only if $f(z)(1 - |z|^2)^p$ is bounded in \mathbb{B} ; f is in \mathcal{B}_0^{p+1} if and only if $(1 - |z|^2)^p f(z) \rightarrow 0$ as $|z| \rightarrow 1^-$.*

Proof. (1). *Necessity.* Let $f \in \mathcal{B}^{p+1}$. Take and fix some $\alpha > p + 1$. It is clear that $Rf \in A_\alpha^1$. Hence, by the proof of [25, Theorem 2.16] and [18, Proposition 1.4.10], we have

$$\begin{aligned} |f(z) - f(0)| &\lesssim \int_{\mathbb{B}} \frac{(1 - |w|^2)^\alpha |Rf(w)| dV(w)}{|1 - \langle z, w \rangle|^{n+\alpha}} \\ &\leq \|f\|_{\mathcal{B}^{p+1}} \int_{\mathbb{B}} \frac{(1 - |w|^2)^{\alpha-p-1}}{|1 - \langle z, w \rangle|^{n+1+(\alpha-p-1)+p}} dV(w) \\ &\simeq \frac{\|f\|_{\mathcal{B}^{p+1}}}{(1 - |z|^2)^p}, \end{aligned}$$

which implies the desired claim.

Sufficiency. If $(1 - |z|^2)^p |f(z)| \leq M$ for some constant $M > 0$, then by [25, Theorem 2.2],

$$f(z) = \int_{\mathbb{B}} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+p}} dV_p(w)$$

Thus, by [18, Proposition 1.4.10] again,

$$\begin{aligned}
 |Rf(z)| &= (n+p+1) \left| \int_{\mathbb{B}} \frac{f(w)}{(1-\langle z, w \rangle)^{n+2+p}} \left(\sum_{k=1}^n z_k \bar{w}_k \right) dV_p(w) \right| \\
 &\lesssim \int_{\mathbb{B}} \frac{|f(w)|(1-|w|^2)^p}{|1-\langle z, w \rangle|^{n+2+p}} dV(w) \\
 &\leq M \int_{\mathbb{B}} \frac{1}{|1-\langle z, w \rangle|^{n+1+p+1}} dV(w) \\
 &\lesssim \frac{1}{(1-|z|^2)^{p+1}},
 \end{aligned}$$

which implies the desired result.

(2). The second assertion follows from (1) and the fact that \mathcal{B}_0^{p+1} and $A_0^{-p}(\mathbb{B})$ are the closure of the polynomials in \mathcal{B}^{p+1} and $A^{-p}(\mathbb{B})$ respectively. \square

From the view of Lemma 7.2, it is clear that we can express $d(f, \mathcal{N}(p, q, s))$ as

$$\inf_{g \in \mathcal{N}(p, q, s)} \|f - g\|_{\mathcal{B}^{\frac{q}{p}+1}}.$$

Now for $f \in H(\mathbb{B})$ and $\varepsilon > 0$, we denote

$$\tilde{\Omega}_\varepsilon(f) = \{z \in \mathbb{B} : |Rf(z)|(1-|z|^2)^{\frac{q}{p}+1} \geq \varepsilon\}.$$

By using the above expression, we have the following result.

Theorem 7.3. *Suppose $p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{n}\}$ and $f \in A^{-\frac{q}{p}}(\mathbb{B}) = \mathcal{B}^{\frac{q}{p}+1}$. Then the following quantities are equivalent:*

- (1) $d_1 = d(f, \mathcal{N}(p, q, s))$;
- (2) $d_4 = \inf\{\varepsilon : \chi_{\tilde{\Omega}_\varepsilon(f)}(z)(1-|z|^2)^{ns} d\lambda(z) \text{ is an } (ns)\text{-Carleson measure}\}$;
- (3)

$$d_5 = \inf \left\{ \varepsilon : \sup_{a \in \mathbb{B}} \int_{\tilde{\Omega}_\varepsilon(f)} |Rf(z)|^p (1-|z|^2)^{p+q} (1-|\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty \right\};$$

(4)

$$d_6 = \inf \left\{ \varepsilon : \sup_{a \in \mathbb{B}} \int_{\tilde{\Omega}_\varepsilon(f)} |\nabla f(z)|^p (1-|z|^2)^{p+q} (1-|\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty \right\};$$

(5)

$$d_7 = \inf \left\{ \varepsilon : \sup_{a \in \mathbb{B}} \int_{\tilde{\Omega}_\varepsilon(f)} |\tilde{\nabla} f(z)|^p (1-|z|^2)^q (1-|\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty \right\}.$$

Proof. (1). $d_1 \lesssim d_4$.

The proof for this part is similar to the proof of $d_1 \lesssim d_2$ in Theorem 7.1. Again, we may assume that $p > 1$. Let ε be a positive number such that $\chi_{\tilde{\Omega}_\varepsilon(f)}(z)(1 - |z|^2)^{ns}d\lambda(z)$ is an (ns) -Carleson measure. Since $f \in \mathcal{B}^{\frac{q}{p}+1}$, we have

$$\sup_{z \in \mathbb{B}} |Rf(z)|(1 - |z|^2)^{\frac{q}{p}+1} < \infty.$$

Take and fix some $\alpha > \max \left\{ \frac{q}{p}, \frac{q+ns-n-1}{p} + 1 \right\}$. It is easy to see that $Rf(z) \in A_\alpha^1$, and hence by [25, Theorem 2.2], we have

$$Rf(z) = \int_{\mathbb{B}} \frac{Rf(w)dV_\alpha(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \quad z \in \mathbb{B}.$$

Since $Rf(0) = 0$, we have

$$Rf(z) = \int_{\mathbb{B}} Rf(w) \left(\frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}} - 1 \right) dV_\alpha(w), \quad z \in \mathbb{B}.$$

It follows that

$$f(z) - f(0) = \int_0^1 \frac{Rf(tz)}{t} dt = \int_{\mathbb{B}} Rf(w) L(z, w) dV_\alpha(w),$$

where the kernel

$$L(z, w) = \int_0^1 \left(\frac{1}{(1 - t\langle z, w \rangle)^{n+1+\alpha}} - 1 \right) \frac{dt}{t}.$$

Let $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = f(0) + \int_{\tilde{\Omega}_\varepsilon(f)} Rf(w) L(z, w) dV_\alpha(w)$$

and

$$f_2(z) = \int_{\mathbb{B} \setminus \tilde{\Omega}_\varepsilon(f)} Rf(w) L(z, w) dV_\alpha(w)$$

We have the following claims.

- $\|f_2\|_{\mathcal{B}^{\frac{q}{p}+1}} \leq C\varepsilon$ **for some constant** $C > 0$.

Since

$$RL(z, w) = \int_0^1 \frac{(n+1+\alpha)\langle z, w \rangle dt}{(1 - t\langle z, w \rangle)^{n+\alpha+2}} = \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}} - 1,$$

we have

$$\begin{aligned}
|Rf_2(z)| &= \left| \int_{\mathbb{B} \setminus \tilde{\Omega}_\varepsilon(f)} Rf(w) RL(z, w) dV_\alpha(w) \right| \\
&\lesssim \varepsilon \int_{\mathbb{B} \setminus \tilde{\Omega}_\varepsilon(f)} (1 - |w|^2)^{\alpha - \frac{q}{p} - 1} \cdot \left(\frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}} - 1 \right) dV(w) \\
&\leq \varepsilon \cdot \left(\int_{\mathbb{B}} \frac{(1 - |w|^2)^{\alpha - \frac{q}{p} - 1}}{|1 - \langle z, w \rangle|^{n+1+\alpha - \frac{q}{p} - 1 + (\frac{q}{p} + 1)}} dV(w) + 1 \right) \\
&\lesssim \frac{\varepsilon}{(1 - |z|^2)^{\frac{q}{p} + 1}},
\end{aligned}$$

which implies the desired claim.

- $f_1 \in \mathcal{N}(p, q, s)$.

By Theorem 5.3, it suffices to show that $|Rf_1(z)|^p (1 - |z|^2)^{p+q+ns} d\lambda(z)$ is a (ns) -Carleson measure. Note that for $z \in \mathbb{B}$, we have

$$\begin{aligned}
|Rf_1(z)| &= \left| \int_{\tilde{\Omega}_\varepsilon(f)} Rf(w) RL(z, w) dV_\alpha(w) \right| \\
&\leq \int_{\tilde{\Omega}_\varepsilon(f)} |Rf(w)| \left(\frac{1}{|1 - \langle z, w \rangle|^{n+1+\alpha}} + 1 \right) dV_\alpha(w) \\
&= I_1 + I_2,
\end{aligned}$$

where

$$I_1 = \int_{\tilde{\Omega}_\varepsilon(f)} \frac{|Rf(w)|}{|1 - \langle z, w \rangle|^{n+1+\alpha}} dV_\alpha(w)$$

and

$$I_2 = \int_{\tilde{\Omega}_\varepsilon(f)} |Rf(w)| dV_\alpha(w).$$

First, we note that by our choice of α , it follows that

$$\begin{aligned}
I_2 &\lesssim \int_{\mathbb{B}} |Rf(w)| (1 - |w|^2)^{\frac{q}{p} + 1} (1 - |w|^2)^{\alpha - \frac{q}{p} - 1} dV(w) \\
&\lesssim \int_{\mathbb{B}} (1 - |w|^2)^{\alpha - \frac{q}{p} - 1} dV(w) \lesssim 1.
\end{aligned}$$

Next, we estimate I_1 . Note that

$$\begin{aligned}
I_1 &\simeq \int_{\tilde{\Omega}_\varepsilon(f)} \frac{|Rf(w)| (1 - |w|^2)^{\frac{q}{p} + 1} (1 - |w|^2)^{\alpha - \frac{q}{p} - 1}}{|1 - \langle z, w \rangle|^{n+1+\alpha}} dV(w) \\
&\lesssim \int_{\mathbb{B}} \frac{(1 - |w|^2)^\alpha}{|1 - \langle z, w \rangle|^{n+1+\alpha}} \cdot \frac{\chi_{\tilde{\Omega}_\varepsilon(f)}(w)}{(1 - |w|^2)^{\frac{q}{p} + 1}} dV(w)
\end{aligned}$$

Let

$$g(w) = \frac{\chi_{\tilde{\Omega}_\varepsilon(f)}(w)}{(1 - |w|^2)^{\frac{q}{p}+1}}$$

and hence we have

$$|g(w)|^p(1 - |w|^2)^{p+q+ns}d\lambda(w) = \chi_{\tilde{\Omega}_\varepsilon(f)}(w)(1 - |w|^2)^{ns}d\lambda(w),$$

which is an (ns) -Carleson measure by our assumption. Now in Theorem 6.5, let $t = 0$ and $b = n + 1 + \alpha$, it is easy to check that

$$t = 0 > n - p - q - ns$$

and

$$b = n + 1 + \alpha > \frac{n + 1}{p'} + \frac{q + ns}{p} + 1.$$

Hence, the operator T with parameter $n + 1 + \alpha$ sends the (ns) -Carleson measure

$$|g(z)|^p(1 - |z|^2)^{p+q+ns}d\lambda(z)$$

to

$$\begin{aligned} & |Tg(z)|^p(1 - |z|^2)^{p+q+ns}d\lambda(z) \\ & \left| \int_{\mathbb{B}} \frac{(1 - |w|^2)^\alpha}{|1 - \langle z, w \rangle|^{n+1+\alpha}} \cdot \frac{\chi_{\tilde{\Omega}_\varepsilon(f)}(w)}{(1 - |w|^2)^{\frac{q}{p}+1}} dV(w) \right|^p (1 - |z|^2)^{p+q+ns}d\lambda(z), \end{aligned}$$

which, by Theorem 6.5, is also an (ns) -Carleson measure. Finally, we have

$$|Rf_1(z)| \leq I_1 + I_2 \lesssim \left| \int_{\mathbb{B}} \frac{(1 - |w|^2)^\alpha}{|1 - \langle z, w \rangle|^{n+1+\alpha}} \cdot \frac{\chi_{\tilde{\Omega}_\varepsilon(f)}(w)}{(1 - |w|^2)^{\frac{q}{p}+1}} dV(w) \right| + 1$$

and it follows that $|Rf_1(z)|^p(1 - |z|^2)^{p+q+ns}d\lambda(z)$ is an (ns) -Carleson measure. Hence, the claim is proved.

Note that $f_1 \in A^{-\frac{q}{p}}(\mathbb{B}) = \mathcal{B}^{\frac{q}{p}+1}$ since $\mathcal{N}(p, q, s) \subseteq A^{-\frac{q}{p}}(\mathbb{B}) = \mathcal{B}^{\frac{q}{p}+1}$. Thus, we have

$$d_1 = d(f, \mathcal{N}(p, q, s)) \lesssim \|f - f_1\|_{\mathcal{B}^{\frac{q}{p}+1}} = \|f_2\|_{\mathcal{B}^{\frac{q}{p}+1}} \lesssim \varepsilon.$$

Finally, by letting ε tends to d_2 , we get the desired result.

(2). $d_4 \leq d_5$.

The proof for this part is almost the same as the proof for $d_2 \leq d_3$ in Theorem 7.1 and hence we omit it here.

(3). $d_5 \leq d_6 \leq d_7$.

This assertion is straightforward from

$$|Rf(z)|(1 - |z|^2) \leq |\nabla f(z)|(1 - |z|^2) \leq |\tilde{\nabla} f(z)|.$$

(see, e.g., [25, Lemma 2.14]).

(4). $d_7 \leq d_1$.

It suffices to show that

$$(d_1, \infty) \subset \left\{ \varepsilon : \sup_{a \in \mathbb{B}} \int_{\tilde{\Omega}_\varepsilon(f)} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty \right\}.$$

Take and fix any $\varepsilon > d_1$, there exists a function $f_1 \in \mathcal{N}(p, q, s)$ such that

$$\|f - f_1\|_{\mathcal{B}^{\frac{q}{p}+1}} < \frac{d_1 + \varepsilon}{2}.$$

Then, by triangle inequality, we have for $z \in \tilde{\Omega}_\varepsilon(f)$,

$$\begin{aligned} |\tilde{\nabla} f_1(z)|(1 - |z|^2)^{\frac{q}{p}} &\geq |Rf_1(z)|(1 - |z|^2)^{\frac{q}{p}+1} \\ &\geq |Rf(z)|(1 - |z|^2)^{\frac{q}{p}+1} - |R(f - f_1)(z)|(1 - |z|^2)^{\frac{q}{p}+1} \\ &\geq \varepsilon - \frac{d_1 + \varepsilon}{2} = \frac{\varepsilon - d_1}{2}. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} &\sup_{a \in \mathbb{B}} \int_{\tilde{\Omega}_\varepsilon(f)} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\ &\lesssim \sup_{a \in \mathbb{B}} \int_{\tilde{\Omega}_\varepsilon(f)} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\ &\leq \left(\frac{2}{\varepsilon - d_1} \right)^p \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\nabla f_1(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) \\ &< \infty, \end{aligned}$$

where in the last inequality, we use Corollary 5.5 and in the first inequality, we use the fact that $1 + \frac{q}{p} > \frac{1}{2}$ and hence by our previous remark,

$$|f(0)| + \sup_{z \in \mathbb{B}} (1 - |z|^2)^{\frac{q}{p}} |\tilde{\nabla} f(z)|$$

becomes an equivalent norm of $\mathcal{B}^{\frac{q}{p}+1}$. Therefore we get the desired inclusion. \square

Corollary 7.4. *Suppose $p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{n}\}$ and $f \in A^{-\frac{q}{p}}(\mathbb{B})$. Then the following conditions are equivalent:*

- (1) f is in the closure of $\mathcal{N}(p, q, s)$ in $A^{-\frac{q}{p}}(\mathbb{B})$.
- (2) $\chi_{\Omega_\varepsilon(f)}(z)(1 - |z|^2)^{ns} d\lambda(z)$ is an (ns) -Carleson measure for every $\varepsilon > 0$;
- (3) $\chi_{\tilde{\Omega}_\varepsilon(f)}(z)(1 - |z|^2)^{ns} d\lambda(z)$ is an (ns) -Carleson measure for every $\varepsilon > 0$;

(4)

$$\sup_{a \in \mathbb{B}} \int_{\Omega_\varepsilon(f)} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda < \infty$$

for every $\varepsilon > 0$;

(5)

$$\sup_{a \in \mathbb{B}} \int_{\tilde{\Omega}_\varepsilon(f)} |Rf(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty.$$

for every $\varepsilon > 0$;

(6)

$$\sup_{a \in \mathbb{B}} \int_{\tilde{\Omega}_\varepsilon(f)} |\nabla f(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty$$

for every $\varepsilon > 0$;

(7)

$$\sup_{a \in \mathbb{B}} \int_{\tilde{\Omega}_\varepsilon(f)} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < \infty$$

for every $\varepsilon > 0$.

For the little-oh version, we denote the distance in $A^{-\frac{q}{p}}(\mathbb{B})$ of f to $\mathcal{N}^0(p, q, s)$ by $d(f, \mathcal{N}^0(p, q, s))$, that is

$$d(f, \mathcal{N}^0(p, q, s)) = \inf_{g \in \mathcal{N}^0(p, q, s)} |f - g|_{\frac{q}{p}}.$$

We have the following result.

Theorem 7.5. *Suppose $p \geq 1, q > 0, s > \max\{0, 1 - \frac{q}{n}\}$ and $f \in A^{-\frac{q}{p}}(\mathbb{B})$. Then the following conditions are equivalent:*

- (1) $e_1 = d(f, A_0^{-\frac{q}{p}}(\mathbb{B}))$;
- (2) $e_2 = d(f, \mathcal{N}^0(p, q, s))$;
- (3)

$$e_3 = \inf\{\varepsilon : \chi_{\Omega_\varepsilon(f)}(z)(1 - |z|^2)^{ns} d\lambda(z) \text{ is a vanishing } (ns)\text{-Carleson measure}\};$$

(4)

$$e_4 = \inf\{\varepsilon : \chi_{\tilde{\Omega}_\varepsilon(f)}(z)(1 - |z|^2)^{ns} d\lambda(z) \text{ is a vanishing } (ns)\text{-Carleson measure}\};$$

(5)

$$e_5 = \inf\left\{\varepsilon : \lim_{|a| \rightarrow 1^-} \int_{\Omega_\varepsilon(f)} |f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) = 0\right\};$$

(6)

$$e_6 = \inf \left\{ \varepsilon : \lim_{|a| \rightarrow 1^-} \int_{\tilde{\Omega}_\varepsilon(f)} |Rf(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) = 0 \right\};$$

(7)

$$e_7 = \inf \left\{ \varepsilon : \lim_{|a| \rightarrow 1^-} \int_{\tilde{\Omega}_\varepsilon(f)} |\nabla f(z)|^p (1 - |z|^2)^{p+q} (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) = 0 \right\};$$

(8)

$$e_8 = \inf \left\{ \varepsilon : \lim_{|a| \rightarrow 1^-} \int_{\tilde{\Omega}_\varepsilon(f)} |\tilde{\nabla} f(z)|^p (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) = 0 \right\}.$$

Proof. Since both $A_0^{-\frac{q}{p}}(\mathbb{B})$ and $\mathcal{N}^0(p, q, s)$ are the closure of all the polynomials in $A^{-\frac{q}{p}}(\mathbb{B})$ and $\mathcal{N}(p, q, s)$ respectively, the equivalence between e_1 and e_2 is obvious.

Moreover, in the proof of Theorem 7.1 and 7.3, by interchanging the role of $d(f, \mathcal{N}(p, q, s))$ to $d(f, \mathcal{N}^0(p, q, s))$, $\sup_{a \in \mathbb{B}}$ to $\lim_{|a| \rightarrow 1^-}$ and applying Lemma 6.7 instead of Lemma 6.5, we can get the equivalence of e_2, e_3, \dots, e_8 . \square

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